UBC-SFU-UVic-UNBC Calculus Exam Solutions, June 2001

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1. (a) Product Rule: $\dfrac{d}{dx}\left(e^x \tan x\right) = e^x \tan x + e^x \sec^2 x$.

(b) Chain Rule: $f'(x) = 2001\left(x^2 + \sqrt{\dfrac{x-\pi}{7}}\right)^{2000}\left[2x + \dfrac{1/7}{2\sqrt{\frac{x-\pi}{7}}}\right]$.

(c) $g'(t) = \dfrac{2}{t}\cos(2\ln t)$, $g''(t) = -\dfrac{4}{t^2}\sin(2\ln t) - \dfrac{2}{t^2}\cos(2\ln t)$, so $g''(1) = -2$.

(d) Quotient Rule: $\dfrac{du}{dx} = \dfrac{\left[2x\cos(x^2)\right]\left(1+\cos^2 x\right) - \sin(x^2)\left[-2\cos x \sin x\right]}{(1+\cos^2 x)^2}$.

2. Parallel lines have equal slopes, so solve $\dfrac{1}{\sqrt{1-x^2}} = \dfrac{2}{\sqrt{3}}$ for $x = \pm\dfrac{1}{2}$.

Two points on the curve have the desired property, namely $\left(\dfrac{1}{2}, \dfrac{\pi}{6}\right)$ and $\left(-\dfrac{1}{2}, -\dfrac{\pi}{6}\right)$.

3. By definition,

$$\begin{aligned} f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(3+h)^2} - \frac{1}{3^2}\right] \\ &= \lim_{h\to 0}\frac{-6-h}{(3+h)^2(3)^2} = -\frac{2}{27}. \end{aligned}$$

4. $1 = yy' = \left(k\sqrt{5x+1}\right)\left(\dfrac{5k}{2\sqrt{5x+1}}\right) = \dfrac{5}{2}k^2$. Given $k > 0$, conclude $k = \sqrt{\dfrac{2}{5}}$.

5. (a) Implicit differentiation gives $3^x(\ln 3) - 2^y(\ln 2)y' = 0$. Plugging in $(x,y) = (2,3)$ gives the slope; the tangent line is

$$y = 3 + \left(\frac{9\ln 3}{8\ln 2}\right)(x-2).$$

(b) The original equation implies $2^y = 3^x - 1$. Substitution in (a) gives

$$y' = \frac{3^x \ln 3}{(3^x - 1)\ln 2} = \frac{\ln 3}{\ln 2\left[1 - 3^{-x}\right]}.$$

As $x \to \infty$, $3^{-x} \to 0$, so $y' \to \dfrac{\ln 3}{\ln 2}$.

Another approach is to solve for $y = \ln(3^x - 1)/\ln 2$ from the given equation and use direct methods to find either the tangent line in (a) or the limit in (b), or both.

6. Write y for the vertical distance from the bottom of the wall to the top of the ladder. Then the desired area is $A = \frac{1}{2}sy$, and Pythagoras gives $s^2 + y^2 = 5^2$. These observations can be combined before or after differentiating, as follows.

(i) Solving for $y = \sqrt{25 - s^2}$ leads to

$$A = \frac{1}{2}s\sqrt{25 - s^2}, \quad \text{so} \quad \frac{dA}{dt} = \frac{dA}{ds}\frac{ds}{dt} = \frac{1}{2}\left(\sqrt{25 - s^2} - \frac{s^2}{\sqrt{25 - s^2}}\right)\left(1 + e^{-s}\right).$$

At the given instant, $y = 3$ and $s = 4$, so $\dfrac{dA}{dt} = -\dfrac{7}{6}\left(1 + e^{-4}\right)$.

(ii) Differentiating $s^2 + y^2 = 25$ gives

$$2s\frac{ds}{dt} + 2y\frac{dy}{dt} = 0, \qquad \text{i.e.,} \qquad \frac{dy}{dt} = -\frac{s}{y}\frac{ds}{dt} = -\frac{s}{y}\left(1 + e^{-s}\right).$$

Differentiating $A = \frac{1}{2}sy$ and substituting from the line above yields

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{ds}{dt}\right)y + \frac{1}{2}s\left(\frac{dy}{dt}\right) = \frac{1}{2}\left(y - \frac{s^2}{y}\right)\left(1 + e^{-s}\right),$$

just as before.

Since $\dfrac{dA}{dt} < 0$, the area is decreasing; its exact rate of change, in m^2/s, is $-\dfrac{7}{6}\left(1 + e^{-4}\right)$.

7. The differential equation implies that $I(x) = I(0)e^{-kx}$. Measuring I in percent gives $I(0) = 100$ and $I(1) = 60$, so $k = -\ln(3/5)$. The desired thickness x, in mm, satisfies

$$1 = I(x) = 100e^{-kx}, \qquad \text{i.e.,} \qquad x = \frac{\ln(100)}{\ln(5/3)} \approx 9.01515.$$

8. Since E is everywhere differentiable, the desired property is equivalent to the assertion that $E'(t) \le 0$ always. We prove this by computing E' with the chain rule, then substituting from the given differential equation:

$$\frac{dE}{dt} = 2y(t)y'(t) + 2y'(t)y''(t) = 2y'(t)\left[y(t) + y''(t)\right] = 2y'(t)\left[-cy'(t)\right] = -2c\left[y'(t)\right]^2.$$

The right side is nonpositive because $c \ge 0$ is given.

9. (a) $v = \dfrac{ds}{dt} = e^{-t}(\cos t - \sin t) > 0 \iff 0 < t < \dfrac{\pi}{4}$ (recall $0 < t < \pi$).

(b) $a = \dfrac{dv}{dt} = -2e^{-t}\cos t > 0 \iff \dfrac{\pi}{2} < t < \pi$ (recall $0 < t < \pi$).

10. (a) The line through $(2,2)$ and $(1,0)$ has slope 2; since it is tangent to the curve $y = f(x)$ at the point $(2,2)$, we have $f'(2) = 2$. (An algebraic solution is also possible: one rearranges Newton's update formula $x_1 = x_0 - f(x_0)/f'(x_0)$ to get $f'(x_0) = 2$.)

(b) Since $f''(x) < 0$ always, the curve $y = f(x)$ is concave down. Hence it lies below its tangents, one of which we have just discussed. In particular, $f(1) < 0$ while $f(2) = 2 > 0$. Existence of f'' implies continuity of f, so the change in sign of f between $x = 1$ and $x = 2$ indicates that it must have a zero between these points.

11. Let r and h be the radius and height of the inscribed cylinder. Similar triangles give

$$\frac{H-h}{r} = \frac{H}{R} \qquad \text{or} \qquad \frac{h}{R-r} = \frac{H}{R}.$$

Only one such equation is needed. Substituting it into the volume formula $V = \pi r^2 h$ leads to one of

$$V = \pi R^2 H \left(\frac{r}{R}\right)^2 \left(1 - \frac{r}{R}\right) \qquad \text{or} \qquad V = \pi R^2 H \left(\frac{h}{H}\right) \left(1 - \frac{h}{H}\right)^2.$$

We illustrate with the first: since $g(x) = x^2(1-x)$ has critical points at $x = 0$ and $x = 2/3$, and the first derivative test shows that the choice $x = 2/3$ gives an absolute maximum for g over $0 \le x \le 1$, it follows that $r/R = 2/3$, i.e., $r = (2/3)R$. The similar-triangles equation above then gives $h = (1/3)H$.

12. The function described here is decreasing and concave up for $x < -2$, decreasing and concave down for $-2 < x < 0$, decreasing and concave up for $0 < x < 2$, and increasing and concave up for $x > 2$. A correct graph should show these features, and mention a (horizontal) point of inflection at $(-2,0)$, another point of inflection at $(0,-2)$, a local and absolute minimum at $(2,y)$ for some unknown $y < -2$, and an x-intercept at the point $(4,0)$.

13. (a) Note that $f'(x) = 12x^3 + 12(k-2)x^2 - 6kx$, $f''(x) = 36x^2 + 24(k-2)x - 6k$. Since $f'(0) = 0$ and $f''(0) = -6k$, the second derivative test implies that $x = 0$ provides a local maximum for f when $k > 0$, and a local minimum (in particular, *no local maximum*) when $k < 0$. To settle the borderline case $k = 0$, write

$$f(x) = 3x^4 - 8x^3 = x^3(3x - 8) \qquad [\text{case } k = 0].$$

Here $f(0) = 0$, but $f(x) < 0$ for small $x > 0$ and $f(x) > 0$ for small $x < 0$, so there is no local maximum at $x = 0$ when $k = 0$. The desired k-values are exactly those satisfying $k > 0$.

(b) Assuming $k > 0$, the local minima are provided by the nonzero roots of

$$f'(x) = 6x\left[2x^2 + 2(k-2)x - k\right].$$

By the quadratic formula, these are $-\dfrac{1}{2}(k-2) \pm \dfrac{1}{2}\sqrt{(k-2)^2+2k}$, and the distance between them is

$$s = \sqrt{(k-2)^2+2k}.$$

(c) Elementary algebra gives

$$s = \sqrt{(k^2-4k+4)+2k} = \sqrt{(k-1)^2+3}.$$

The choice $k=1$ clearly minimizes this over all real k; the restriction to $k>0$ is required to justify using the form of s derived in part (b).