

Key for the Calculus Challenge Exam in 00-2

1. (a) We can divide out $x - 1$ from both numerator and denominator obtaining:

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^4 - 1} = \lim_{x \rightarrow 1} \frac{x^2 + x + 1}{x^3 + x^2 + x + 1} = \frac{\lim_{x \rightarrow 1} (x^2 + x + 1)}{\lim_{x \rightarrow 1} (x^3 + x^2 + x + 1)} = \frac{3}{4}$$

One can use l'Hospital's rule to reach the same conclusion.

$$(b) \lim_{x \rightarrow 0^+} \frac{x + 2}{2 + \sqrt{x}} = \frac{\lim_{x \rightarrow 0^+} (x + 2)}{\lim_{x \rightarrow 0^+} (2 + \sqrt{x})} = \frac{(\lim_{x \rightarrow 0^+} x) + 2}{2 + \lim_{x \rightarrow 0^+} \sqrt{x}} = \frac{0 + 2}{2 + 0} = 1.$$

- (c) This is a more subtle problem; if the absolute value signs are removed, then the limit does not exist. Note that $\sqrt{1 - \cos 2x}/|x|$ is an even function so it is sufficient to evaluate $\lim_{x \rightarrow 0^+} (\sqrt{1 - \cos 2x}/x)$. By l'Hospital's rule we have

$$\lim_{x \rightarrow 0^+} (\sqrt{1 - \cos 2x}/x)^2 = \lim_{x \rightarrow 0^+} \frac{1 - \cos 2x}{x^2} = \lim_{x \rightarrow 0^+} \frac{\sin 2x}{x} = 2.$$

Hence $\lim_{x \rightarrow 0^+} \sqrt{1 - \cos 2x}/x = \sqrt{2}$.

Another approach is to use the identity $1 - \cos 2x = 2 \sin^2 x$ which gives

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{|x|} = \lim_{x \rightarrow 0} \frac{\sqrt{2 \sin^2 x}}{|x|} = \lim_{x \rightarrow 0} \frac{\sqrt{2} \sin x}{x} = \sqrt{2}.$$

2. The relevant theorem, the IVT, says that, if the function $f(x)$ is continuous on the closed interval $[a, b]$ of \mathbb{R} , and

$$f(a) < r < f(b) \quad \text{or} \quad f(b) < r < f(a),$$

then there exists $c \in (a, b)$ such that $f(c) = r$.

In the present case note that $p(x) = 2x^5 - 2x^4 + 2x^2 - 2x - 1$ is a polynomial, hence continuous, and in particular continuous on the closed interval $[-1/2, 0]$. Also,

$$p(0) = -1 < 0 < 5/16 = p(-1/2).$$

So the theorem applies to $p(x)$ with $a = -1/2$, $b = 0$, and $r = 0$.

3. The correct matching is:

function	(a)	(b)	(c)	(d)	(e)
derivative	(3)	(4)	(1)	(5)	(2)

$$4. \quad (a) \quad \frac{d}{dx} \left(\frac{x}{1+x} \right) = \frac{-x}{(1+x)^2} \frac{d}{dx} (1+x) + \frac{1}{1+x} = \frac{1}{(1+x)^2}$$

$$(b) \quad \frac{d}{dx} (\ln(\sqrt[3]{x})) = \frac{1}{\sqrt[3]{x}} \frac{d}{dx} (\sqrt[3]{x}) = \frac{1}{\sqrt[3]{x}} \frac{1}{3} (x)^{-2/3} = \frac{1}{3x}$$

We can also begin by rewriting $\ln(\sqrt[3]{x})$ as $(1/3) \ln x$.

$$(c) \quad \frac{d}{dx} \left(x \sin^{-1}(1/x) \right) = \sin^{-1}(1/x) + \frac{x}{\sqrt{1-(1/x)^2}} \frac{d}{dx} (1/x) \\ = \sin^{-1}(1/x) - \frac{1}{\sqrt{x^2-1}}.$$

5. We are given the equations:

$$\begin{cases} V &= \frac{1}{3}\pi r^2 h \\ 1 &= h^2 + r^2. \end{cases}$$

We have to maximize V . Eliminating r we get

$$V = \frac{1}{3}(1-h^2)h \\ \frac{dV}{dh} = \frac{1}{3}(1-3h^2) = -\left(h - \frac{1}{\sqrt{3}}\right) \left(h + \frac{1}{\sqrt{3}}\right).$$

Thus the critical points for V are $h = \pm 1/\sqrt{3}$. In terms of the construction of the cone only values of h in $(0, 1)$ are meaningful. By inspection of the equation above, $dV/dh > 0$ for $0 < h < 1/\sqrt{3}$, and $dV/dh < 0$ for $1/\sqrt{3} < h < 1$. Thus the maximum value of V is attained when $h = 1/\sqrt{3}$.

When $h = 1/\sqrt{3}$, we have $r^2 = 1 - h^2 = 2/3$. So

$$V_{\max} = \frac{1}{3}\pi(1-h^2)h = \frac{2\pi}{9\sqrt{3}}.$$

6. Differentiating implicitly the equation

$$\sin(\pi xy) = \frac{2}{3}(x+y)$$

we get

$$\pi \cos(\pi xy) \left[y + x \frac{dy}{dx} \right] = \frac{2}{3} \left(1 + \frac{dy}{dx} \right) \quad (1)$$

Setting $x = 1$, $y = 1/2$ we see that $\cos(\pi xy) = 0$, and so from (1), at $(1, 1/2)$, $dy/dx = -1$.

Differentiating (1) implicitly we get

$$-\pi^2 \sin(\pi xy) \left[y + x \frac{dy}{dx} \right]^2 + \pi \cos(\pi xy) \left[2 \frac{dy}{dx} + x \frac{d^2 y}{dx^2} \right] = \frac{2}{3} \frac{d^2 y}{dx^2}. \quad (2)$$

Setting $x = 1$, $y = 1/2$ in (2), we have

$$-\pi^2 [(1/2) - 1]^2 = \frac{2}{3} \frac{d^2 y}{dx^2},$$

which says that $d^2 y/dx^2 = -(3\pi^2)/8$.

7. Let $f(x) = \tan^{-1} x$. By definition, $f(x)$ is the unique θ in $-\pi/2 < x < \pi/2$ such that $\tan \theta = x$. Hence $f(1) = \pi/4$. Note that

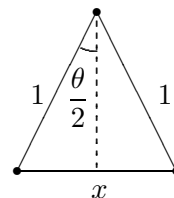
$$f'(1) = \left(\frac{d}{dx} \tan^{-1} x \right) \Big|_{x=1} = \left(\frac{1}{1+x^2} \right) \Big|_{x=1} = 1/2.$$

Taking 1 as base point the linear approximation is

$$f(1.1) \approx f(1) + (.1)f'(1) = \pi/4 + (.1)(.5) = \pi/4 + .05.$$

8. From the diagram on the right we see that

$$x = 2 \sin \frac{\theta}{2}.$$



We are given that, when $\theta = \pi/3$, then $\frac{d\theta}{dt} = 1$.

Differentiating with respect to t and then setting $\theta = \pi/3$ we get

$$\frac{dx}{dt} = \left(\cos \frac{\theta}{2} \right) \frac{d\theta}{dt} = \left(\cos \frac{\pi}{6} \right) (1) = \frac{\sqrt{3}}{2}.$$

There are less convenient ways of expressing x in terms of θ . For example the law of cosines gives

$$x^2 = 1^2 + 1^2 - 2(1 \cdot 1) \cos \theta = 2(1 - \cos \theta).$$

We get the same expression by using the formula for the distance between the points $(\cos \theta, \sin \theta)$ and $(1, 0)$ in the cartesian plane. Rather than taking the square root it is easier to differentiate the equation as it stands:

$$2x \frac{dx}{dt} = 2 \sin \theta \frac{d\theta}{dt}.$$

Setting $\theta = \pi/3$ gives $x = 1$ and the same result as before since $\sin \pi/3 = \sqrt{3}/2$.

9. (a) In evaluating the limit as $x \rightarrow \infty$ of a quotient of polynomials one can ignore all except the highest powers of x in the numerator and denominator. Thus

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 4x + 3}{1 - x^2} = \lim_{x \rightarrow \infty} \frac{2x^2}{-x^2} = -2$$

and similarly $\lim_{x \rightarrow -\infty} \frac{2x^2 - 4x + 3}{1 - x^2} = -2$. Thus $y = -2$ is an asymptote which the graph approaches both as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

- (b) We have:

$$\lim_{x \rightarrow (-1)^-} \frac{2x^2 - 4x + 3}{1 - x^2} = \lim_{x \rightarrow (-1)^-} \frac{2x^2 - 4x + 3}{1 - x} \lim_{x \rightarrow (-1)^-} \frac{1}{1 + x} = \frac{9}{2}(-\infty) = -\infty$$

$$\lim_{x \rightarrow (-1)^+} \frac{2x^2 - 4x + 3}{1 - x^2} = \lim_{x \rightarrow (-1)^+} \frac{2x^2 - 4x + 3}{1 - x} \lim_{x \rightarrow (-1)^+} \frac{1}{1 + x} = \frac{9}{2}(\infty) = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{2x^2 - 4x + 3}{1 - x^2} = \lim_{x \rightarrow 1^-} \frac{2x^2 - 4x + 3}{1 + x} \lim_{x \rightarrow 1^-} \frac{1}{1 - x} = \frac{1}{2}(\infty) = \infty$$

$$\lim_{x \rightarrow 1^+} \frac{2x^2 - 4x + 3}{1 - x^2} = \lim_{x \rightarrow 1^+} \frac{2x^2 - 4x + 3}{1 + x} \lim_{x \rightarrow 1^+} \frac{1}{1 - x} = \frac{1}{2}(-\infty) = -\infty.$$

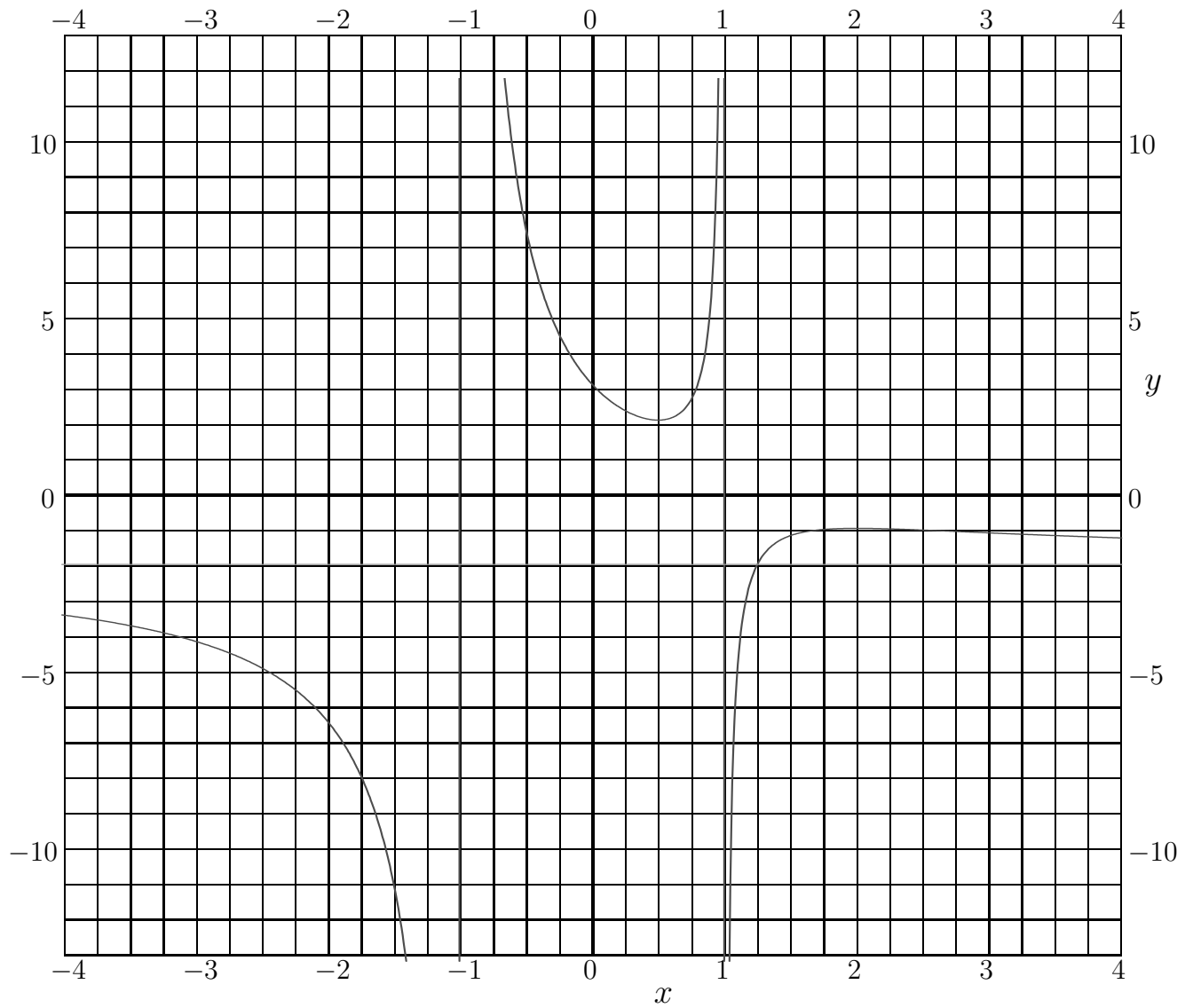
- (c) By inspection of the expression for $f'(x)$ we see that $f'(x) > 0$ for $x \in (1/2, 1)$ and $x \in (1, 2)$, and that $f'(x) < 0$ for $x \in (-\infty, -1)$, $(-1, 1/2)$, and $x \in (2, \infty)$. Hence we have:

$f(x)$ is increasing on $[1/2, 1)$ and $(1, 2]$

$f(x)$ is decreasing on $(-\infty, -1)$, $(-1, 1/2]$, and $[2, \infty)$.

- (d) The critical points are $x = 1/2$ and $x = 2$. Based on the information in part (c) it is clear that f has a local minimum at $x = 1/2$ and a local maximum at $x = 2$. This is the first derivative test. One could apply the second derivative test if one wished.

(e) Here is the graph:



The key points to be noted are:

- $y = -2$ is a horizontal asymptote
- $x = -1$ and $x = 1$ are vertical asymptotes
- f has a local minimum at $(1/2, 2)$ and a local maximum at $(2, -1)$
- the only point at which f crosses $y = -2$ is $(5/4, -2)$.

10. (a) The speed is diminishing by 20 ft/sec each second. Thus the speed is 0 after 4 seconds.
- (b) Integrating once we get the equation

$$\frac{ds}{dt} = -20t + C.$$

Measuring time from the instant when braking begins we have $C = 80$, the initial (uniform) speed. Integrating again gives

$$s = -10t^2 + 80t + C'.$$

Since s and t are measured from the braking point, $C' = 0$. Thus $s = -10t^2 + 80t$ and in the 4 seconds it takes to bring the car to a stop the car travels $-160 + 320 = 160$ ft.

- (c) There will be an accident if the car travels more than 40 ft before braking begins. At 80 ft/sec the car travels 40 ft in half a second. So the driver has half a second to react.
11. (a) The differential equation leads to the conclusion that

$$A = Ie^{-kt}$$

where I is the initial amount.

Since the half-life is 5 years, we have $e^{-5k} = 1/2$. Taking natural logarithms,

$$k = \frac{1}{5} \ln 2.$$

- (b) For the site to be habitable we need the amount of radioactive material to be reduced to $I/7$. So we need

$$e^{-kt} = 1/7$$

Taking natural logarithms and substituting for k , we get $t = (5 \ln 7) / \ln 2$ years.

12. (a) $-\pi \leq \theta \leq 3\pi$.

(b) $x = (\theta \cos \theta) / \pi$, $y = (\theta \sin \theta) / \pi$.

(c) It is easy to obtain that $\left. \frac{dx}{d\theta} \right|_{\theta=2\pi} = 1$ and $\left. \frac{dy}{d\theta} \right|_{\theta=2\pi} = 2\pi$.

So the slope of curve at $\theta = 2\pi$ is $(dy/d\theta) / (dx/d\theta) = 2\pi$.

The equation of the tangent line at $(2, 0)$ is $y = 2\pi(x - 2)$.