Rebookings in Acute Healthcare: A Queueing Theory Model

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1 Introduction

This project primarily focuses on the effect of rebooking booked elective (BE) admissions to hospitals. The principal aim is to determine how various access thresholds for BE admissions relate to the number of hospital beds. The most applicable access threshold is the percentage of patients who are rebooked. To encapsulate the necessary information and dynamics of a hospital, we will formulate a queueing model of the situation.

Initially, we investigate an analytic approach to the problem using a number of methods: direct, iteration, and Taylor series expansion. Using these results, with given parameter values, we numerically determine access thresholds for BE admissions. Finally, we compare these results with simulations and perform a sensitivity analysis of model parameters.

2 Model

Patients who desire surgery join the Surgical Wait List (SWL). These surgeries are non-emergency and may include knee, hip-joint, or cataract surgery. It is possible that a patient may leave SWL for many reasons. A patient may require surgery of higher priority, find surgery elsewhere, or die.

Now, each day, a certain number patients from SWL are booked for surgery, and these patients are BE admissions. If a hospital administrator does not foresee an inpatient bed becoming available soon for a given patient, the administrator cancels this patient. These cancelled patients are in fact rebooked for surgery, and hence rejoin SWL.

Patients who are not cancelled obtain surgery and first go to Post Anesthetic Recovery. Afterwards, they obtain an inpatient bed, recover, and leave the hospital. Figure 1 is a diagram of the entire situation.

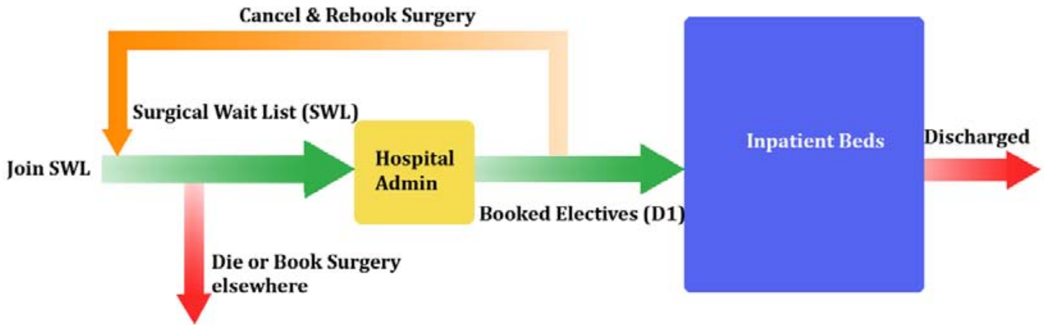

FIGURE 1: Network diagram of hospital

Given these conditions, we make the following assumptions:

(i) SWL and BE admission are FIFO queues

(ii) Patients arrive to SWL by a Poisson process with rate λ

(iii) Reneging from SWL is exponentially distributed with mean $1/\alpha$

(iv) One administrator books surgeries by a Poisson process with rate ν

(v) BE admissions retrial deterministically with parameter τ

(vi) Retrialed BE admissions enter SWL queue by FIFO discipline

(vii) A patient recovers from a bed by a Poisson process with rate μ

Assumptions (ii), (iv), and (vii) are justified by Kingman's Theorem since we are only concerned with heavy traffic situations. Data from the SWL strongly supports assumption (iii). While assumptions (i) and (vi) may be unrealistic, we retain them for simplicity purposes.

The correctness of assumption (v) is not verifiable. This premise is largely based on intuition - hospital administrators should not have largely fluctuating planning times from day to day. Since no data is readily available to estimate τ, it is important that we conduct a sensitivity analysis on this parameter.

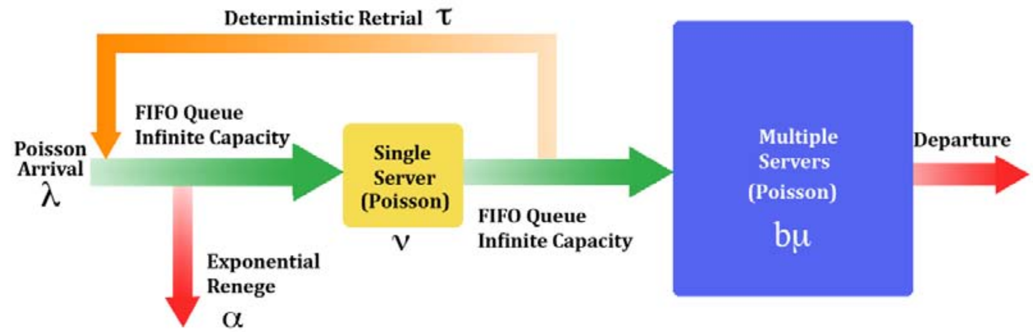

FIGURE 2: Network diagram of queueing system

Figure 2 is a network diagram of the situation reformulated in terms of queueing theory. Note b is the number of inpatient beds. Our primary goal is to achieve a retrial probability of less than 2%.

3 Analytic Results

Consider the model as two connected systems:

System 1 (S1): SWL and the hospital administrator

System 2 (S2): BE admissions and inpatient beds

Hence, S1 is an M/M/1 queue with exponential reneging, and S2 is an M/M/c queue with deterministic reneging. Observe that:

1. the output process of S1 is the input process of S2
2. the input process of S1 includes the reneging from S2

We assume that the reneging process of S2 is a Poisson process with parameter λ_R. This immediately implies that S1 has Poisson input process with parameter $\lambda + \lambda_R$. We must therefore solve for λ_R in terms of all other parameters. Using this assumption and our two key observations above, we approach the problem by three different methods.

3.1 Direct Approach

By Burke's Theorem, the exit process of S1 is Poisson with parameter $E_1(\lambda_R)$. One can verify that

$$E_1(\lambda_R) = (\lambda + \lambda_R) \cdot \mathscr{P}(\lambda + \lambda_R)$$

where $\mathscr{P}(\overline{\lambda})$ is the probability a customer acquires service in S1 with input parameter $\overline{\lambda}$.

Similarly, by assumption, the reneging process of S2 is Poisson with parameter $E_2(\lambda_R)$. From our previous observation, S2 has input process equal to $E_1(\lambda_R)$. Therefore,

$$E_2(\lambda_R) = E_1(\lambda_R) \cdot \mathscr{Q}\left(E_1(\lambda_R)\right)$$

where $\mathscr{Q}(\overline{\lambda})$ is the probability a customer reneges from S2 with input parameter $\overline{\lambda}$.

By definition of λ_R, we have the following equation

$$\lambda_R = E_2(\lambda_R)$$

It remains to solve this equation for λ_R. The individual formulae for $\mathscr{P}(\overline{\lambda})$ and $\mathscr{Q}(\overline{\lambda})$ are relatively straightforward to derive (see Appendices A & B).

3.2 Taylor Expansion

Let $f(\lambda_R) = E_2(\lambda_R) - \lambda_R$. From §3.1, we desire the zeros of f. To do this, we compute the order 1 Taylor expansion of f about $\lambda_R = 0$. Given the equations in §3.1, we have

$$\begin{aligned}
f(0) &= \lambda \cdot \mathscr{P}(\lambda) \cdot \mathscr{Q}\left(\lambda \mathscr{P}(\lambda)\right) \\
f'(0) &= \left[\mathscr{P}(\lambda) + \lambda \cdot \mathscr{P}'(\lambda)\right] \cdot \left[\mathscr{Q}\left(\lambda \mathscr{P}(\lambda)\right) + \lambda \mathscr{P}(\lambda) \cdot \mathscr{Q}'\left(\lambda \mathscr{P}(\lambda)\right)\right] - 1
\end{aligned}$$

Since $\lambda_R << 1$, we have

$$f(\lambda_R) = f(0) + f'(0)\lambda_R + \mathcal{O}(\lambda_R^2)$$

Hence, the zeroes of f may be well approximated by $\lambda_R \approx -\dfrac{f(0)}{f'(0)}$.

3.3 Iterative Method

We define the following recurrence relation.

$$\begin{aligned} \lambda_R^{(0)} &= 0 \\ \lambda_R^{(k)} &= E_2(\lambda_R^{(k-1)}) \quad k \geq 1 \end{aligned}$$

We therefore propose that

$$\lim_{k\to\infty} \lambda_R^{(k)} = \lambda_R$$

for regions where the limit exists.

4 Numerical Results

Symbolically speaking, any of the above analytic methods yield intractable or excessively complicated results. However, given values for all other parameters, one can efficiently compute λ_R by any of the above methods. We restrict our numerical computations to methods described in §3.1 and §3.3.

Below is a table of parameter values we used which are typical of a medium sized hospital.

$\lambda = 0.997$/hr	
$\nu = 0.83$/hr	$\mu = 0.2\overline{6}$/day
$\alpha = 2$/year	$\tau = 8$ hours

Note that λ, ν were slightly perturbed to avoid singularities in E_1 and E_2. We define the *effective arrival rate* of our queueing system to be $\lambda + \lambda_R$.

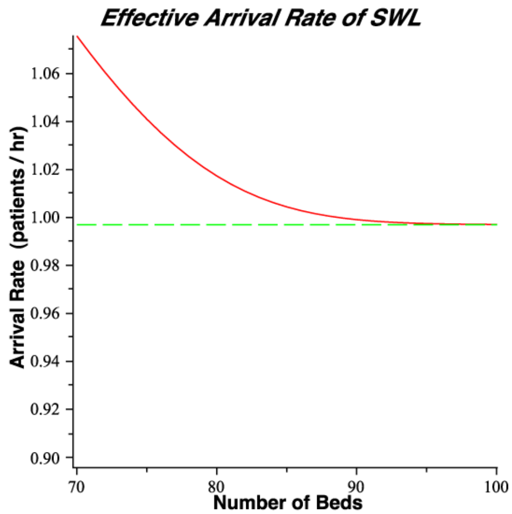

FIGURE 3: Effective Arrival Rate versus Number of Beds
(computed using the Direct Method)

As expected, in Figure 3, we see $\lambda_R \to 0$ as $b \to \infty$. We also computed the effective arrival rate using the Iterative Method and for $b = 70 \ldots 100$, the maximum absolute difference between the two methods was 6.6×10^{-10}. This strongly suggests that the Iterative Method is valid. Henceforth, however, all plots utilize the effective arrival rate computed from the Direct Method.

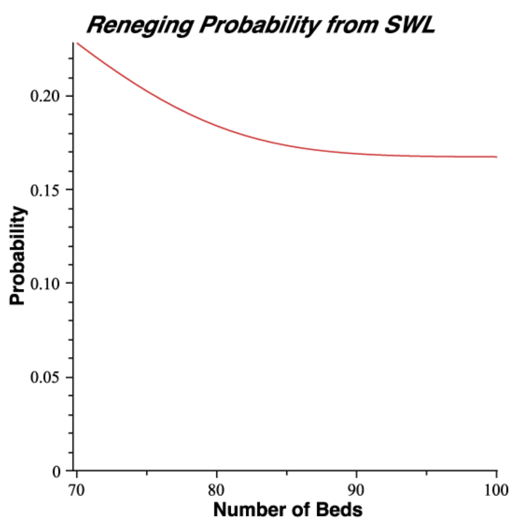

FIGURE 4: Renege Probability versus Number of Beds

In Figure 4, it seems that as $b \to \infty$, there is a limiting renege probability. This is due to the fact $\nu < \lambda$, which implies there is heavy traffic in the SWL queue. Hence, the SWL queue is always sufficiently long; thus, there will exist some steady amount of reneging even as $\lambda_R \to 0$.

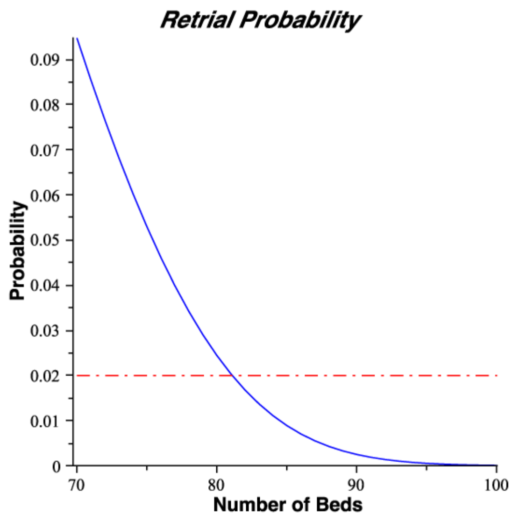

FIGURE 5: Retrial Probability versus Number of Beds

In Figure 5, it is clear that the retrial probability goes to zero as $b \to \infty$. Also, we can see that the desired 2% threshold is close to 81 beds. To confirm this analytic result, we execute some simulations.

5 Simulations

Using parameter values identical to that of §4, we ran computer simulations for $b = 79, 80, \ldots, 84$. For each value of b, we performed 25 trials to obtain statistical validity for the results. For the probability axis in each figure below, we also include $2 \times$ mean standard deviation.

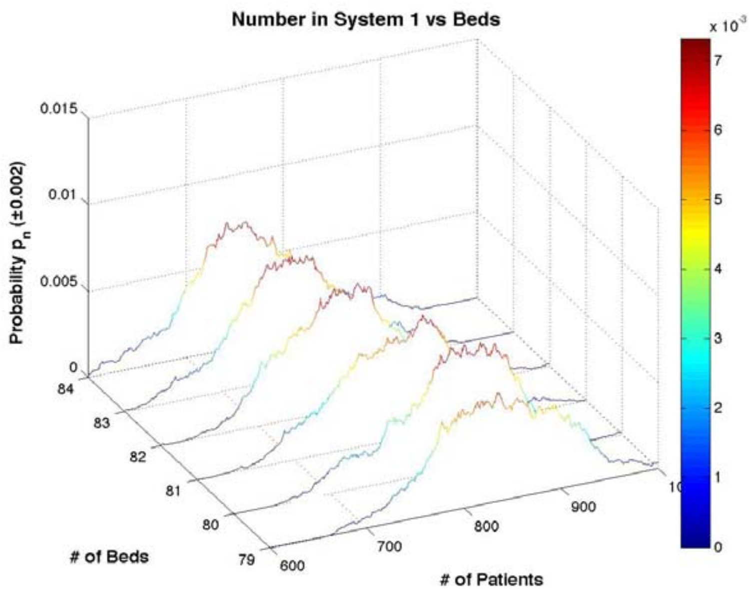

FIGURE 6: Number in System 1

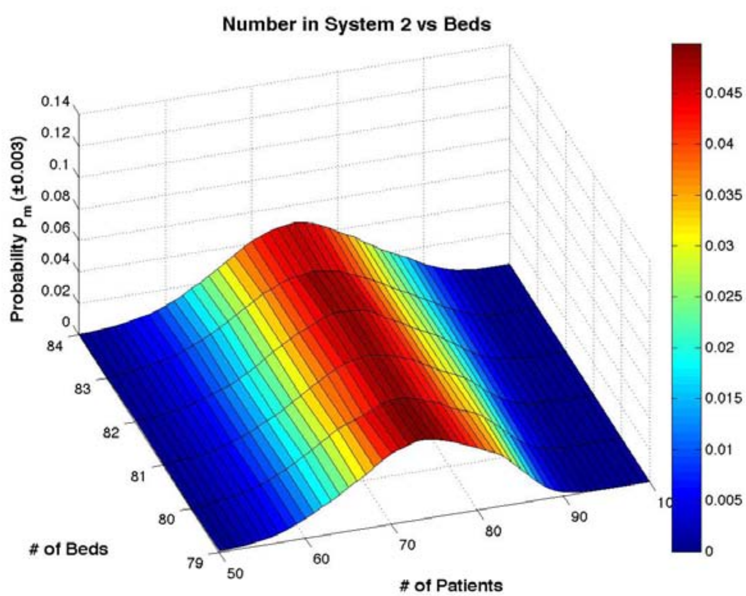

FIGURE 7: Number in System 2

For Figures 8 and 9, ϵ is the maximum absolute difference between the simulation and analytic curve.

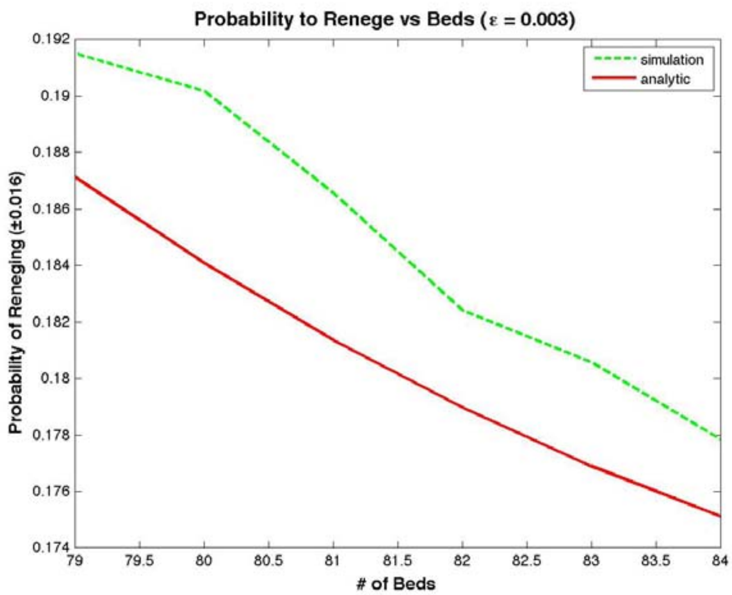

Figure 8: Reneging Probability

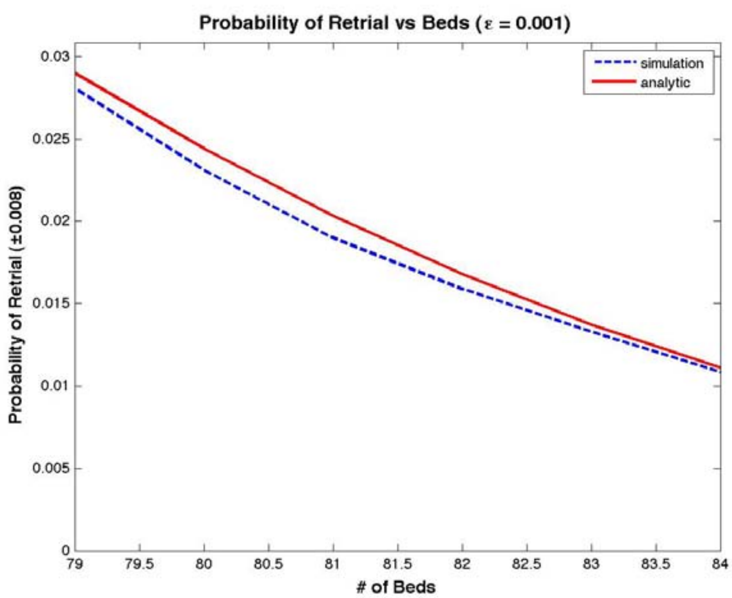

Figure 9: Retrial Probability assuming served by hospital administrator

In Figure 8 and 9, $\epsilon = 0.3\%$ and $\epsilon = 0.1\%$ respectively. Both of these values are very small, and demonstrate strong evidence that our numerical results can accurately predict the necessary number of beds to achieve the 2% threshold. We note, however, that the standard deviation is relatively large $-$ 1.6% and 0.8% for Figures 8 and 9 respectively. This would suggest that more trials per b value are necessary to obtain a stronger result.

Despite no numerical or analytic comparison, we include wait time distributions from the simulations in Figures 10 and 11 below for reader interest.

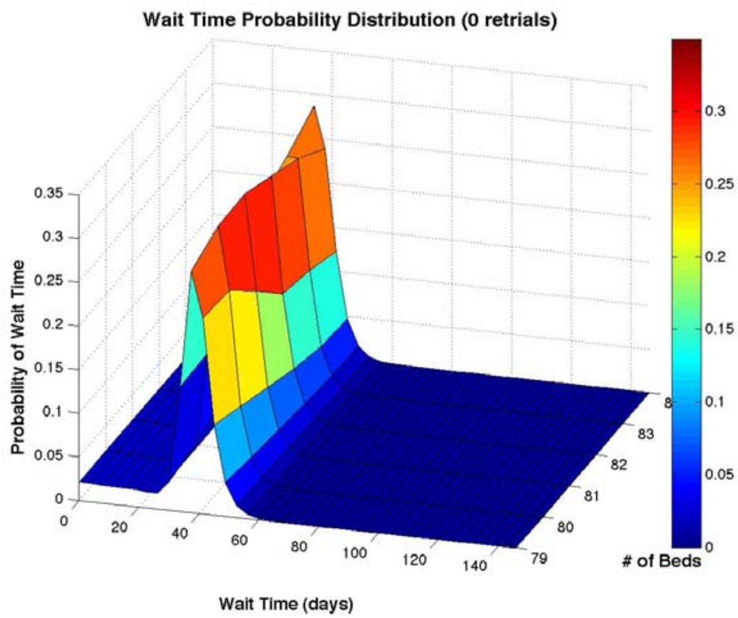

FIGURE 10: Wait Time Distribution for Patients with 0 retrials

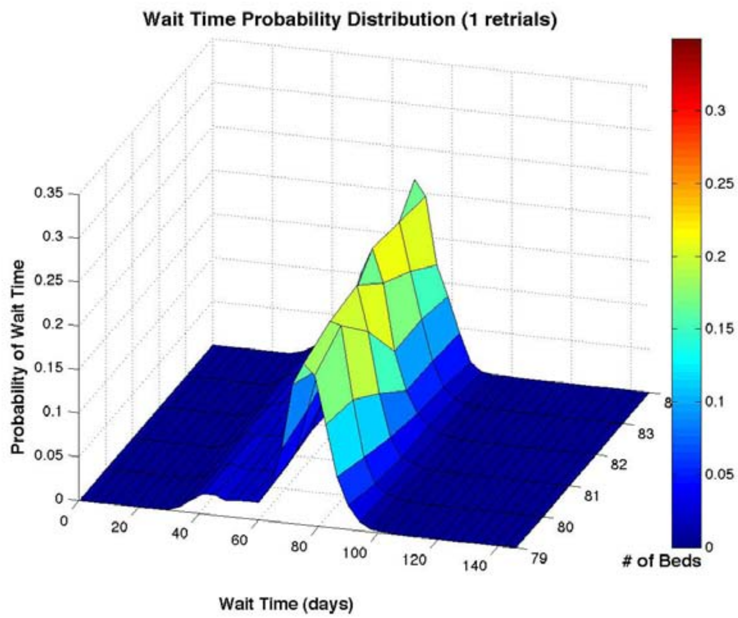

Figure 11: Wait Time Distribution for Patients with exactly 1 retrial

Notice that with one additional retrial, the peak of the distribution essentially shifts by 40 days. This is very likely due to the queue discipline of retrials. Since a retrial returns to the back of SWL, in effect, one restarts the entire wait period.

6 Sensitivity Analysis

For the calculations and simulations in §3 and §4, we used $\tau = 8$ hours for the cancellation window. As noted in §2, the cancellation window is not measurable. It is merely an intuitive notion for the amount of time a hospital administrator requires to plan for BE admissions. Thus, it is pertinent that we conduct a sensitivity analysis of the parameter τ.

Due to computational restrictions, we confine our analysis to the numerical methods of §3, and exclude simulations. Again, we utilize parameter values as specified in §3. Moreover, we range $\tau = 4, 5, \ldots, 16$ for $b = 70, 71, \ldots, 100$ and plot the retrial probability in Figure 12. In Figure 13, for each b value, we compute the maximum difference in retrial probability over all specified τ values.

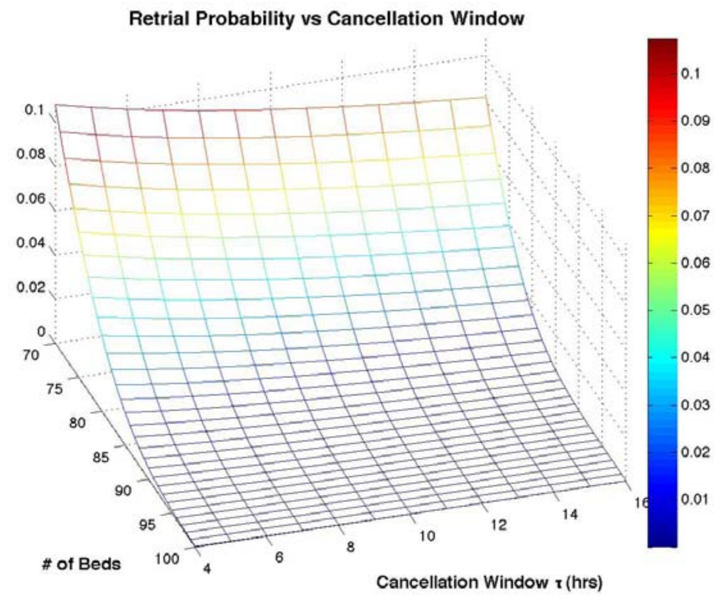

FIGURE 12: Retrial Probability versus Cancellation Window

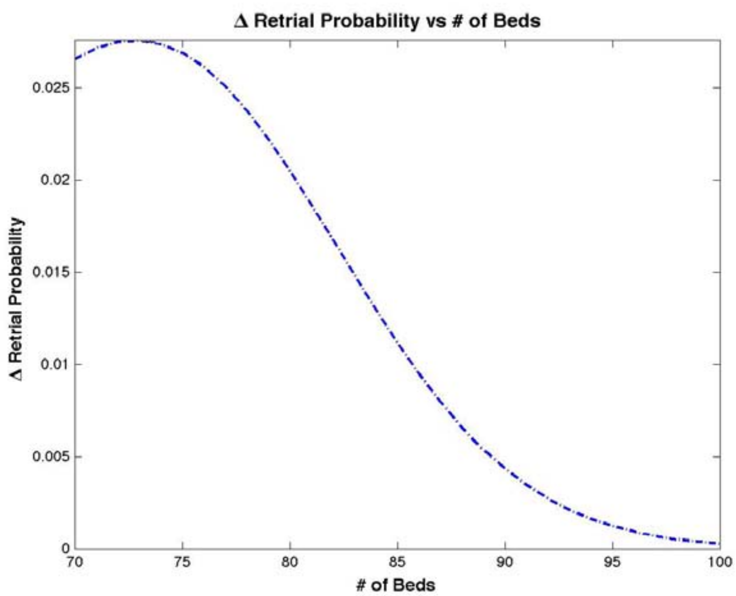

FIGURE 13: Maximum difference in Retrial Probability versus # Beds

Considering Figure 13 alone, we see that the retrial probability changes very little with respect to τ, at most 3%. Moreover, as $b \to \infty$, the effect of τ

appears to be almost completely negligible. On the other hand, if we also consider Figure 12, we observe that, for our region of interest ($b \approx 80$), the retrial probability can range from 2% to 4% over τ. This can be quite a large change in our predictions, which suggests that it is perhaps better to underestimate the value of τ when providing values of b that obtain the 2% threshold.

7 Conclusion

This project is part of the initial study for a healthcare model aimed to minimize rebooking BE admissions by selecting an appropriate number of inpatient beds. Our analytic and numerical investigation of the queueing system proved to be efficient and successful when compared with computer simulations. However, the moderate sensitivity of the cancellation window does spark some concern, and merits further analysis.

For future work, it is proposed to

- verify simulation and numerical results with available data
- include different queue disciplines for SWL and retrials
- incorporate Emergency Department stream
- examine numerical computation of wait time distributions

I would like to thank Dr. Alexander Rutherford, Dr. Bojan Ramadanovic, and Pouya Bastani for all their helpful advice and comments during this entire project.

APPENDIX

A Exponential Reneging (M/M/1)

Let λ be the arrival rate, ν be the service rate, and α be the reneging rate. For simplicity, let $\tilde{\nu} = \nu/\alpha$ and $\tilde{\lambda} = \lambda/\alpha$.

- number in the system probabilities

$$p_n = \begin{cases} \tilde{\lambda}^n \cdot \dfrac{\Gamma(\tilde{\nu})}{\Gamma(n+\tilde{\nu})} \cdot p_0 & (n > 0) \\ \\ \left[1 + \tilde{\lambda}^{1-\tilde{\nu}} \cdot e^{\tilde{\lambda}} \cdot \gamma(\tilde{\nu}, \tilde{\lambda})\right]^{-1} & (n = 0) \end{cases}$$

- probability of <u>not</u> reneging given n in system upon arrival

$$\beta_n = \frac{\nu}{\nu + n\alpha}$$

- probability of <u>not</u> reneging

$$\mathscr{P} = \frac{\tilde{\nu} \cdot e^{\tilde{\lambda}} \cdot \gamma(\tilde{\nu}, \tilde{\lambda})}{\tilde{\lambda}^{\tilde{\nu}} + \tilde{\lambda} \cdot e^{\tilde{\lambda}} \cdot \gamma(\tilde{\nu}, \tilde{\lambda})}$$

- expected number in the system

$$\mathbb{E}[N] = \frac{(\tilde{\lambda} - \tilde{\nu} + 1) \cdot e^{\tilde{\lambda}} \cdot \gamma(\tilde{\nu}, \tilde{\lambda}) + \tilde{\lambda}^{\tilde{\nu}}}{e^{\tilde{\lambda}} \cdot \gamma(\tilde{\nu}, \tilde{\lambda}) + \tilde{\lambda}^{\tilde{\nu}-1}}$$

B Deterministic Reneging (M/M/c)

Let λ be the arrival rate, μ be the service rate, and τ be the maximum wait time. For simplicity, let $\rho = \lambda/\mu$.

- number in the system probabilities

$$p_n = \begin{cases} \dfrac{\lambda^n}{\mu^{n-c} c!} p_0 \cdot \displaystyle\prod_{k=1}^{n-c} (c\mu + \beta_{c+k})^{-1} & (n \geq c) \\[2ex] \dfrac{\rho^n}{n!} p_0 & (0 < n \leq c) \\[2ex] \left[\dfrac{e^{\rho} \cdot \Gamma(c+1, \rho)}{c!} + \dfrac{\rho^{c+1} \cdot (e^{\alpha(\rho - c)} - 1)}{(\rho - c) c!} \right]^{-1} & (n = 0) \end{cases}$$

- probability of reneging given $c + n$ in system upon arrival

$$\beta_{c+n} = \frac{(c\mu\tau)^n \cdot e^{-c\mu\tau}}{\tau \cdot \gamma(n, c\mu\tau)} \qquad (n \geq 1)$$

- probability of reneging

$$\mathscr{Q} = \frac{(\rho - c) \cdot \rho^c \cdot \exp\{\mu\tau(\rho - c)\}}{(\rho - c) \cdot e^{\rho} \cdot \Gamma(c+1, \rho) + \rho^{c+1} \cdot (\exp\{\mu\tau(\rho - c)\} - 1)}$$