Simon Fraser University
Department of Mathematics
Burnaby Campus

MATH 251, Summer 2006
Midterm 2
July 5th, 2006, 8:30 – 9:20

Last Name (please print):	Solution
First Name (please print):	~~ ~~
SFU Email ID:	~~ ~~
Instructor:	Dr. G. Tanoh

Instructions:

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.
2. Fill in the above box.
3. This exam contains 7 pages with a total of 6 questions. Once the exam begins please check to make sure your exam is complete.
4. **Show all your work and explain your answers clearly.**
5. If you run out of space in a problem, use the space on the back of the previous page and clearly indicate where the solution continues.
6. **Only** scientific, non-programmable calculators with no differentiation and integration capabilities are allowed.
7. No book, paper, or device, other than the usual writing instruments, this booklet and an acceptable calculator, shall be within reach of a student during the examination.
8. During the examination, speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.

Do not write in this table!	
Question	**Marks**
1	**/6**
2	**/8**
3	**/7**
4	**/7**
5	**/5**
6	**/7**
Total	**/40**

Question 1. The trajectory of a particle in space is given by the position vector $r(t) = (2 + 3t + 3t^2)\mathbf{i} + (4t + 4t^2)\mathbf{j} - (6\cos t)\mathbf{k}$.

(a) [4] Find the normal and tangent components of the acceleration vector.

$$r'(t) = (3+6t)i + (4+8t)j + (6\sin t)k, \quad |r'(t)| = \left[25(1+2t)^2 + 36\sin^2 t\right]^{1/2}$$

$$r''(t) = 6i + 8j + (6\cos t)k, \quad r'(t)\cdot r''(t) = 50(1+2t) + 18\sin 2t$$

$$r'(t)\times r''(t) = \begin{vmatrix} i & j & k \\ 3+6t & 4+8t & 6\sin t \\ 6 & 8 & 6\cos t \end{vmatrix}$$

$$= 24\left[(1+2t)\cos t - 2\sin t\right]i - 18\left[(1+3t)\cos t - 2\sin t\right]j$$

$$|r'(t)\times r''(t)| = 6\left[16\left((1+2t)\cos t - 2\sin t\right)^2 + 9\left((1+3t)\cos t - 2\sin t\right)^2\right]^{1/2}$$

$$a_T = \frac{r'(t)\cdot r''(t)}{|r'(t)|} = \frac{50(1+2t) + 18\sin 2t}{\left[25(1+2t)^2 + 36\sin^2 t\right]^{1/2}}$$

$$a_N = \frac{|r'(t)\times r''(t)|}{|r'(t)|} = \frac{6\left[16\left((1+2t)\cos t - 2\sin t\right)^2 + 9\left((1+3t)\cos t - 2\sin t\right)^2\right]^{1/2}}{\left[25(1+2t)^2 + 36\sin^2 t\right]^{1/2}}$$

(b) [2] What is the speed and acceleration vector of the particle at the point $(2, 0, -6)$?

The point $(2,0,6)$ corresponds to $t = 0$ $(r(0) = \langle 2, 0, -6\rangle)$

speed $= |r'(0)| = \sqrt{25} = 5$

Acceleration vector $= r''(0) = 6i + 8j + (6\cos 0)k$

$= 6i + 8j + 6k$

Question 2. In each of the following cases find the limit, if it exists, or show that the limit does not exist:

(a) [4] $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$.

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^4-y^2}{x^4+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^4-y^2}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4} = 1$$

Since $-1 \neq 1$ the limit does not exist

(b) [4] $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2}$.

$$\lim_{\substack{(x,y)\to 0\\ x=y}} \frac{x^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3}{2x^2} = \lim_{x\to 0} \frac{x}{2} = 0$$

We want to find $\delta > 0$ such that

$$\frac{|x|\,x^2}{x^2+y^2} = \left|\frac{x^3}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

$x^2 \le x^2+y^2 \Rightarrow |x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$. We have

$\frac{x^2}{x^2+y^2} \le 1 \Rightarrow \frac{|x|\,x^2}{x^2+y^2} \le |x| \le \sqrt{x^2+y^2}$. So if $\delta = \varepsilon$ and let $0 < \sqrt{x^2+y^2} < \delta$, we get $\left|\frac{x^2}{x^2+y^2} - 0\right| \le |x| \le \sqrt{x^2+y^2} < \delta = \varepsilon$

Therefore $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$

Question 3. Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $w = f(x, y)$.

(a) [3] Show that $\dfrac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta$ and $\dfrac{1}{r}\dfrac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta$. Applying chain Rule

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = f_x(-r\sin\theta) + f_y\, r\cos\theta$$
$$= r(-f_x\sin\theta + f_y\cos\theta)$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta$$

(b) [2] Solve the equations in part (a) to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial \theta$.

$$\begin{cases} f_x\cos\theta + f_y\sin\theta = \dfrac{\partial w}{\partial r} \\ -f_x\sin\theta + f_y\cos\theta = \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} \end{cases}$$

A linear system of equations with unknowns f_x and f_y

$$f_x = \cos\theta\,\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$f_y = \sin\theta\,\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}$$

(c) [2] Show that $(f_x)^2 + (f_y)^2 = \left(\dfrac{\partial w}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial w}{\partial \theta}\right)^2$. From (a) we have

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = (f_x\cos\theta + f_y\sin\theta)^2 + (-f_x\sin\theta + f_y\cos\theta)^2$$
$$= (f_x)^2\cos^2\theta + 2f_xf_y\cos\theta\sin\theta + (f_y)^2\sin^2\theta$$
$$+ (f_x)^2\sin^2\theta - 2f_xf_y\sin\theta\cos\theta + (f_y)^2\cos^2\theta$$
$$= (f_x)^2(\cos^2\theta + \sin^2\theta) + (f_y)^2(\sin^2\theta + \cos^2\theta)$$
$$= (f_x)^2 + (f_y)^2$$

Question 4. At the point $(1,2)$, the function $f(x,y)$ has a derivative of 2 in the direction toward $(2,2)$ and a derivative of -2 in the direction toward $(1,1)$.

(a) [4] Find $f_x(1,2)$ and $f_y(1,2)$.

The vector from $(1,2)$ to $(2,2)$ is $\langle 2-1, 2-2\rangle = \langle 1, 0\rangle = i$

the vector from $(1,2)$ to $(1,1)$ is $\langle 1-1, 1-2\rangle = \langle 0, -1\rangle = -j$

i and j are unit vectors, so $\nabla f(1,2)\cdot i = 2$ and $\nabla f(1,2)\cdot(-j) = -2$

$$\Rightarrow \begin{cases} f_x(1,2)\cdot(1) + f_y(1,2)(0) = 2 \\ f_x(1,2)\cdot(0) + f_y(1,2)(-1) = -2 \end{cases} \Rightarrow \begin{cases} f_x(1,2) = 2 \\ f_y(1,2) = 2 \end{cases}$$

(b) [2] Find the derivative of f at $(1,2)$ in the direction toward the point $(4,6)$. From (a) $\nabla f(1,2) = \langle 2,2\rangle$

The vector from $(1,2)$ to $(4,6)$ is $\langle 4-1, 6-2\rangle = \langle 3,4\rangle$

the unit vector is $\dfrac{\langle 3,4\rangle}{|\langle 3,4\rangle|} = \dfrac{\langle 3,4\rangle}{\sqrt{9+16}} = \frac{3}{5} i + \frac{4}{5} j$

$$\nabla f(1,2)\cdot\langle \tfrac{3}{5}, \tfrac{4}{5}\rangle = f_x \tfrac{3}{5} + f_y \tfrac{4}{5} = 2\tfrac{3}{5} + 2\tfrac{4}{5} = \frac{14}{5}$$

(c) [1] What is the largest value that the derivative of f can have at the point $(1,2)$?

The largest value is $|\nabla f(1,2)| = \sqrt{2^2+2^2} = 2\sqrt{2}$

Question 5. [5] Find the absolute maximum and minimum values of $f(x, y) = 3 + xy - x - 2y$ on the closed triangular region D with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$.

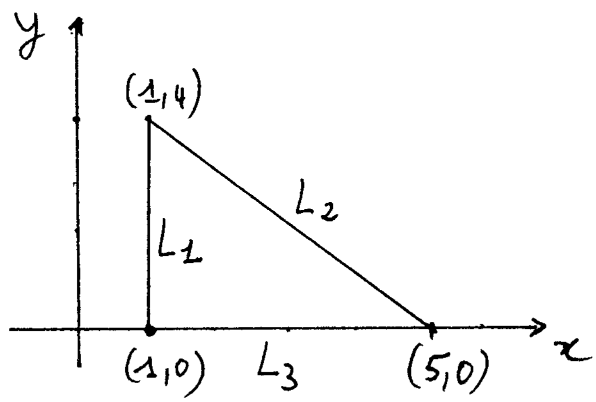
Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = y-1$, $f_y = x-2$ and setting $f_x = f_y = 0$ gives $(2,1)$ as a critical point, where $f(2,1) = 1$

Along L_1: $x=1$ and $f(1,y) = 2-y$ for $0 \leq y \leq 4$, a decreasing function in y, so the maximum value is $f(1,0) = 2$ and the minimum value is $f(1,4) = -2$.

Along L_2: $y=0$ and $f(x,0) = 3-x$ for $1 \leq x \leq 5$, a decreasing function in x, so the maximum is $f(1,0)=2$ and the minimum is $f(5,0) = -2$.

Along L_3: $y = 5-x$ and $f(x, 5-x) = x^2+6x-7 = (x-3)^2+2$ for $1 \leq x \leq 5$, which has a maximum at $x=3$ where $f(3,2)=2$ and a minimum at both $x=1$ and $x=5$, where $f(1,4)=f(5,0)=2$.

Thus the absolute maximum of f on D is $f(1,0)=f(3,2)=2$ and the absolute minimum is $f(1,4) = f(5,0) = -2$



Question 6. [7] Find the ~~maximum and~~ minimum values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

Using Lagrange Multipliers we have

$$\left\{\begin{array}{l} 2x = \lambda + \mu \\ 2y = 2\lambda + 3\mu \\ 2z = 3\lambda + 9\mu \\ x + 2y + 3z = 6 \\ x + 3y + 9z = 9 \end{array}\right. \iff \left\{\begin{array}{l} x = \frac{1}{2}(\lambda+\mu) \\ y = \lambda + \frac{3}{2}\mu \\ z = \frac{3}{2}\lambda + \frac{9}{2}\mu \\ \frac{1}{2}(\lambda+\mu) + 2(\lambda + \frac{3}{2}\mu) + 3(\frac{3}{2}\lambda + \frac{9}{2}\mu) = 6 \\ \frac{1}{2}(\lambda+\mu) + 3(\lambda + \frac{3}{2}\mu) + 9(\frac{3}{2}\lambda + \frac{9}{2}\mu) = 9 \end{array}\right.$$

$$\left\{\begin{array}{l} x = \frac{1}{2}(\lambda+\mu) \\ y = \lambda + \frac{3}{2}\mu \\ z = \frac{3}{2}\lambda + \frac{9}{2}\mu \\ 7\lambda + 17\mu = 6 \\ 34\lambda + 91\mu = 18 \end{array}\right. \quad \left\{\begin{array}{l} x = \frac{1}{2}\left(\frac{240}{59} - \frac{78}{59}\right) = \frac{81}{59} \\ y = \frac{240}{59} - \frac{3}{2}\cdot\frac{78}{59} = \frac{123}{59} \\ z = \frac{3}{2}\,\frac{240}{59} - \frac{9}{2}\,\frac{78}{59} = \frac{9}{59} \\ \lambda = \frac{240}{59}, \quad \mu = -\frac{78}{59} \end{array}\right.$$

the minimum value is

$$f\left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right) = \frac{81^2 + 123^2 + 9^2}{59^2} = \frac{21771}{3481}$$

$$\simeq 6.254$$