[4] 1. (a) Give the equation of the plane tangent to the surface $z = 4-x^2-y^2$ at the point $(1,1,2)$.

$$z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$f_x = -2x$
$f_y = -2y$

$$z - 2 = -2(x-1) + -2(y-1)$$

$$2x + 2y + z = 6$$

either answer o.k.

[2] (b) What is the direction of most rapid increase of the function $f(x,y) = 4 - x^2 - y^2$ at the point $(3,2)$?

$$\nabla f(x,y) = \langle -2x, -2y \rangle$$
$$\nabla f(3,2) = \langle -6, -4 \rangle$$

$\therefore$ most rapid increase in direction

$\langle -3, -2 \rangle$ [or any scalar multiple]

[4] (c) In what direction(s) is the directional derivative of $f(x,y) = 4 - x^2 - y^2$ at the point $(3,2)$ equal to zero?

$$0 = D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} \qquad \vec{u} = \langle \alpha, \beta \rangle$$

$$\Rightarrow \langle -6, -4 \rangle \cdot \langle \alpha, \beta \rangle = 0$$

$$\Rightarrow -6\alpha - 4\beta = 0$$

$$\Rightarrow 3\alpha = -2\beta$$

directions $\langle 1, -\frac{3}{2} \rangle$ and $\langle -1, \frac{3}{2} \rangle$

or $\langle 2, -3 \rangle$ and $\langle -2, 3 \rangle$

[or scalar mult.]

[8] 2. Use the method of Lagrange multipliers to find the closest point(s) on the surface $x^2yz = 1$ to the origin.

minimize $\sqrt{x^2+y^2+z^2}$ on $x^2yz-1=0$

$\therefore$ take $f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = x^2yz-1 = 0$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 2xyz \\ x^2z \\ x^2y \end{bmatrix} = \lambda \nabla g$$

[3] for setup

$\therefore$ $2x^2 = 2\lambda x^2yz$
$4y^2 = 2\lambda x^2yz$
$4z^2 = 2\lambda x^2yz$

[3] for work

$\therefore$ $2x^2 = 4y^2 = 4z^2 \Rightarrow z = \pm y$, $x = \pm\sqrt{2}\,y$

$x^2yz = \pm 2y^2yy = 1$

$\Rightarrow y^4 = \pm\frac{1}{2}$ $\therefore$ $y^4 = \frac{1}{2}$, $y = \pm 2^{-\frac{1}{4}}$

[2] for sol'ns

$\therefore$ 4 closest points: $\left(\pm 2^{\frac{1}{4}}, 2^{-\frac{1}{4}}, 2^{-\frac{1}{4}}\right)$

and $\left(\pm 2^{\frac{1}{4}}, -2^{-\frac{1}{4}}, -2^{-\frac{1}{4}}\right)$

[8] 3. (a) Find and classify the critical points of

$$f(x, y) = 3x - 3y + x^2 - xy + 2y^2.$$

$$\left.\begin{array}{l} f_x(x,y) = 3 + 2x - y = 0 \\ f_y(x,y) = -3 - x + 4y = 0 \end{array}\right\} \begin{array}{l} \Rightarrow y = 3 + 2x \\ \Rightarrow -3 - x + 4(3+2x) = 0 \\ \Rightarrow x = -\frac{9}{7}, \ y = \frac{3}{7} \end{array}$$

[4] for critical pt.

$\therefore$ one critical point at $\left(-\frac{9}{7}, \frac{3}{7}\right)$

[4] for local min & some justification

$A = f_{xx} = 2 > 0$

$B = f_{xy} = -1$

$C = f_{yy} = 4$

$\Delta = AC - B^2 = 8 - 1 = 7 > 0$

$\therefore \left(-\frac{9}{7}, \frac{3}{7}\right)$ is a local minimum point & also an absolute min. pt.

[6] (b) The area of an ellipse is given by $A = \pi ab$, where a and b are the lengths of the semiaxes and the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Consider an cylinder of height $h = 3$, above an ellipse in the x-y plane with $a = 2$ and $b = 4$. How fast is the volume V of the cylinder increasing or decreasing if $\frac{dh}{dt} = 1$, $\frac{da}{dt} = 2$, and $\frac{db}{dt} = -\frac{1}{2}$?

$V = \pi a b h$

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} + \frac{dV}{da}\frac{da}{dt} + \frac{dV}{db}\frac{db}{dt}$$

[3] for chain rule

$$= \pi ab \cdot 1 + \pi bh \cdot 2 + \pi ah \cdot \left(-\frac{1}{2}\right)$$

$$= \pi\left[ab + 2bh - \frac{ah}{2}\right]$$

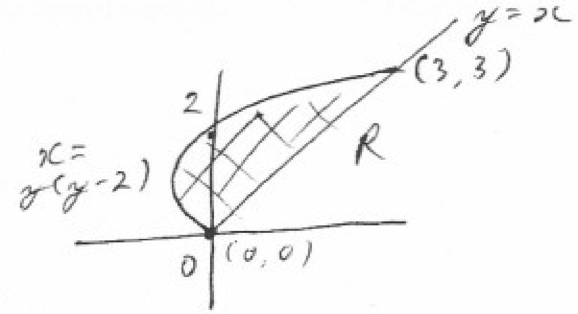
$$= \pi[8 + 24 - 3]$$

$$\boxed{= 29\pi}$$

[3] for work & sol'n

[2] for accurate sketch

(sketch not required for full marks)



$y = y(y-2) \Rightarrow y = 0$
or $y - 2 = 1 \Rightarrow y = 3$

[8] 4. (a) Use double integration to find the volume under the surface $z = x^2 + y$ over the region bounded by $x = y^2 - 2y$ and $y = x$.

[2] for 1st integration

$$V = \iint_R (x^2+y)\, dA = \int_0^3 \int_{y^2-2y}^{y} (x^2+y)\, dx\, dy$$

$$= \int_0^3 \left[\frac{x^3}{3} + xy\right]_{y^2-2y}^{y} dy \qquad = \int_0^3 * \, dy$$

(see extra page)

$$= -\frac{729}{7} + 243 - \frac{4 \cdot 243}{5} + \frac{81}{2} + 27$$

$$= \frac{837}{70}$$

[4] (b) Find

$$\int_0^\pi \int_0^\pi (e^x + \cos y)\, dx\, dy$$

$$= \int_0^\pi \left[e^x + x\cos y\right]_0^\pi dy$$

$$= \int_0^\pi \left[e^\pi + \pi\cos y - 1\right] dy$$

[2]

$$= \left[y(e^\pi - 1) + \pi \sin y\right]_0^\pi$$

[1]

$$= \pi(e^\pi - 1)$$

[1]

4. (a)

$$* = \left[\frac{y^3}{3} + y^2\right] - \left[\frac{(y^2-2y)^3}{3} + (y^2-2y)y\right]$$

$$= \frac{y^3}{3} + y^2 - y^3 + 2y^2 - \frac{1}{3}\left[y^6 - 6y^5 + 12y^4 - 8y^3\right]$$

$$= -\frac{y^6}{3} + 2y^5 - 4y^4 + 2y^3 + 3y^2$$

$$\therefore \int_0^3 * \, dy = \left[-\frac{y^7}{3\cdot 7} + \frac{2y^6}{6} - \frac{4}{5}y^5 + \frac{2}{4}y^4 + \frac{3}{3}y^3\right]_0^3$$

FULL MARKS FOR THIS LINE or further, with adequate work shown

$$= \frac{-3^7}{3\cdot 7} + \frac{2\cdot 3^6}{6} - \frac{4}{5}3^5 + \frac{1}{2}3^4 + 3^3$$

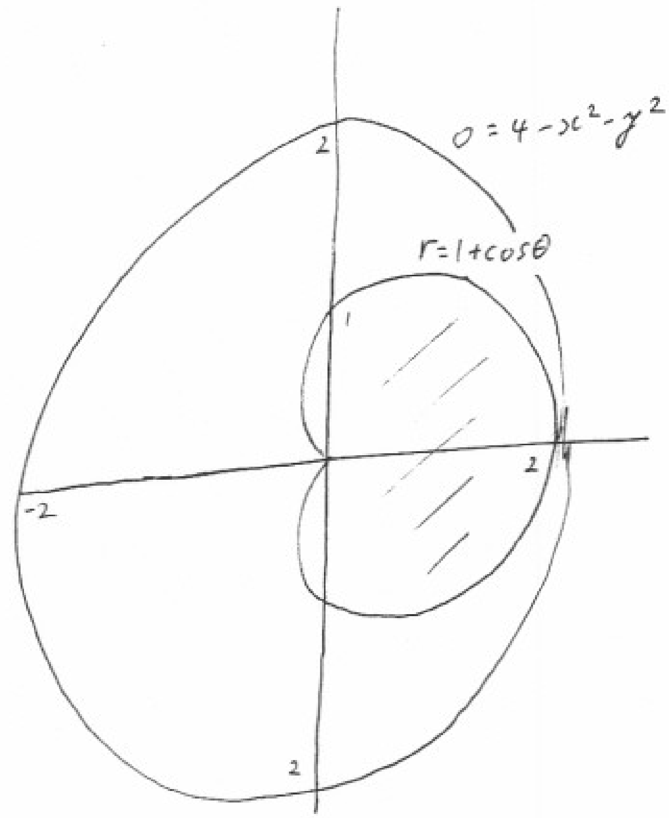
$$= -\frac{729}{7} + 243 - \frac{4}{5}\cdot 243 + \frac{81}{2} + 27$$

[6] 5. Find the integral expression in polar coordinates for the volume of intersection of the cardiod cylinder $r = 1 + \cos\theta$ and the sphere $z^2 = 4 - x^2 - y^2$. Do not carry out the integration.

vol inside $z = \pm\sqrt{4-r^2}$ & inside $r = 1 + \cos\theta$ [2]

$$V = 2\int_0^{2\pi}\int_0^{1+\cos\theta} \sqrt{4-r^2}\, r\,dr\,d\theta$$ [4]

by symmetry.



[2] for sketch

(not necessary for full marks)