1a) (4 marks) By making a transformation to polar coordinates, show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0.$$

$x = r\cos\theta,\ y = r\sin\theta$

As $(x,y) \to (0,0)$, $r \to 0$ (independently of θ)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \cancel{r^2}\left(\frac{\cos^2\theta - \sin^2\theta}{\cancel{r}}\right)$$

$\cos^2\theta - \sin^2\theta \leq 1$

$$\lim_{r\to 0} r(\cos^2\theta - \sin^2\theta) = 0$$

1b) (4 marks) By making the substitution $y = mx$, demonstrate that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+xy}$$

does **not** exist.

Put $y = mx$. Then $\dfrac{xy}{x^2+y^2+xy} = \dfrac{m}{1+m^2+m}$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2+xy} = \dfrac{m}{1+m^2+m}$ along $y = mx$

put $m = 1$, limit $= \dfrac{1}{3}$

put $m = -1$, limit $= +1$

$\therefore$ No unique value that $\dfrac{xy}{x^2+y^2+xy}$ approaches as $(x,y)\to(0,0)$ in all directions.

$\therefore$ There is no limit

2) (8 marks) By using the differential of a suitably defined function, approximate the number $(\sqrt{17} + \sqrt{99})^2$.

Define $f(x,y) = (\sqrt{x} + \sqrt{y})^2$

Find the differential of f, df:

$$f_x = 2(\sqrt{x}+\sqrt{y}) \times \frac{1}{2} x^{-1/2} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$$

$$f_y = 2(\sqrt{x}+\sqrt{y}) \times \frac{1}{2} y^{-1/2} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{y}}$$

$$f(x+dx, y+dy) - f(x,y) \approx df = f_x(x,y)\,dx + f_y(x,y)\,dy$$

put $x = 16$, $y = 0$

then $dx = 1$, $dy = -1$

$$f(x+dx, y+dy) = f(17,99) \approx f(16,100) + f_x(16,100)\,dx + f_y(16,100)\,dy$$

$$= (4+10)^2 + \frac{4+10}{4} \times 1 + \frac{4+10}{10} \times -1$$

$$= 198.1$$

3) Let $w = \sqrt{x^2+y^2+z^2}$ and let $x = 3e^t \sin s$, $y = 3e^t \cos s$ and $z = 4e^t$.

(i) (2 marks) Name the independent and the intermediate variables.

Independent: s, t

Intermediate: x, y, z

(ii) (6 marks) Use the chain rule to find $\partial w/\partial t$ and $\partial w/\partial s$, expressing your answers as functions of t and s and simplifying as much as possible.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} \times 3e^t \sin s + \frac{y}{\sqrt{x^2+y^2+z^2}} \times 3e^t \cos s + \frac{z}{\sqrt{x^2+y^2+z^2}} \times 4e^t$$

$$= \frac{9e^{2t}\sin^2 s + 9e^{2t}\cos^2 s + 16e^{2t}}{\sqrt{9e^{2t}\sin^2 s + 9e^{2t}\cos^2 s + 16e^{2t}}} = \frac{25e^{2t}}{\sqrt{25e^{2t}}} = 5e^t$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= \frac{x \times 3e^t \cos s - y \times 3e^t \sin s + z \times 0}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{9e^t \sin s \cos s - 9e^t \sin s \cos s}{5e^t} = 0$$

4) (8 marks) Find the maximum and minimum values of $f(x,y) = xy$ subject to the constraint $4x^2 + 9y^2 = 36$.

Use $\nabla g = \lambda \nabla f$ where $f = xy$, $g = 4x^2+9y^2-36$.

$\nabla f = (y, x)$, $\nabla g = (8x, 18y)$

$\therefore$ $y = 8\lambda x$, $x = 18\lambda y$

$\therefore$ $\lambda = \dfrac{y}{8x} = \dfrac{x}{18y} \Rightarrow 18y^2 = 8x^2,\ 9y^2 = 4x^2$

Also $4x^2+9y^2 = 36 \Rightarrow 8x^2 = 36$

$\therefore$ $x = \pm\dfrac{3}{\sqrt{2}}$. $y^2 = \dfrac{4}{9}x^2$, $y = \pm\dfrac{2}{3}x = \pm\sqrt{2}$.

Must have all 4 possibilities, i.e. critical points are

$\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$, $\left(\frac{-3}{\sqrt{2}}, -\sqrt{2}\right)$, $\left(\frac{-3}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(\frac{3}{\sqrt{2}}, -\sqrt{2}\right)$

P_1 P_2 P_3 P_4

Curve $g = 0$ is bounded (an ellipse).

$$\begin{cases} \text{Maximum of } f = 3 & \text{at } P_1 \text{ and } P_2 \\ \text{minimum of } f = -3 & \text{at } P_3 \text{ and } P_4 \end{cases}$$

(Subject to the constraint $g(x,y) = 0$)

5) (8 marks) Find and classify the critical points of $f(x, y) = x^4 + y^4 - 4xy$.

Critical points are at $f_x = f_y = 0$ because the derivatives of $f(x,y)$ are continuous to all orders

$f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$.

$x^3 = y$ and $y^3 = x$

$\therefore$ $x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0$

$\therefore$ $x = 0$, $+1$ and -1.

$\therefore$ $y = x^3 = 0, +1$ and -1.

The critical points are $P_1(0,0)$, $P_2(1,1)$ and $P_3(-1,-1)$

Now calculate f_{xx}, f_{yy} and f_{xy} :-

$f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = f_{yx} = -4$.

Classify :

Point P_1 $f_{xx}f_{yy} - f_{xy}^2 = 0 - 16 < 0$ Saddle point

Point P_2 $\left.\begin{array}{l} f_{xx}f_{yy} - f_{xy}^2 = 144 - 16 > 0 \\ \text{and } f_{xx} > 0 \end{array}\right\}$ local minimum

Point P_3 $\left.\begin{array}{l} f_{xx}f_{yy} - f_{xy}^2 = 144 - 16 > 0 \\ \text{and } f_{xx} > 0 \end{array}\right\}$ local minimum