Math 251 - Midterm 1
Summer 2006

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Question 1.

(a) $\vec{AB} = \langle 0-(-2), 1-2, -1-0 \rangle = \langle 2, -1, -1 \rangle$

$\vec{AC} = \langle -1-(-2), 2-2, -2-0 \rangle = \langle 1, 0, -2 \rangle$

$$\vec{AB} \times \vec{AC} \begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 0 & -2 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} j$$

$$+ \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} k = (2-0)i - (-4+1)j + (0+1)k$$

$$\vec{AB} \times \vec{AC} = 2i + 3j + k = \langle 2, 3, 1 \rangle$$

(b) The area of triangle ABC is $\frac{1}{2} |\vec{AB} \times \vec{AC}|$

$|\vec{AB} \times \vec{AC}| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14}$.

So the area of triangle ABC is $\frac{\sqrt{14}}{2}$

(c) A normal vector to the plane is $n = \vec{AB} \times \vec{AC}$

We have $n = \langle 2, 3, 1 \rangle$. With the point $A(-2, 2, 0)$ an equation of the plane is

$$2(x+2) + 3(y-2) + 1 \cdot (z-0) = 0$$

$$2x + 3y + z = 2$$

Question 2 If $a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$ and $c = \langle c_1, c_2, c_3 \rangle$, then

$$a \cdot (b \times c) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

$$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 = (a \times b) \cdot c$$

Question 3 (a) We find a particular point on the line of intersection by setting $z = 0$, we have $2x + 2y = 3$ and $x + 2y = 2$, whose solution is $x = 1$, $y = \frac{1}{2}$. So the point $(1, \frac{1}{2}, 0)$ lies on the ligne of intersection. A vector parallel to this line is given by $v = n_1 \times n_2$, where

$n_1 = \langle 2, 2, -1 \rangle$ is a normal vector to the plane $2x + 2y - z = 3$, and $n_2 = \langle 1, 2, 1 \rangle$ is a normal vector to the plane $x + 2y + z = 2$.

$$v = \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} k$$

$$= (2+2)i - (2+1)j + (4-2)k$$

$$= 4i - 3j + 2k$$

So ~~a~~ parametric equations are

$$x = 1 + 4t \qquad y = \tfrac{1}{2} - 3t \qquad z = 2t$$

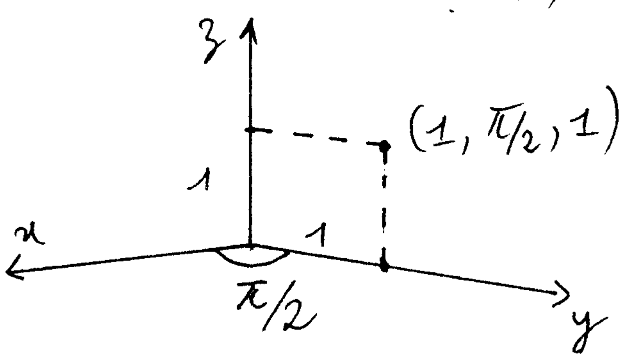
<u>Question 3</u> (b) If θ is the angle between the planes, we have

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1|\cdot|n_2|} = \frac{\langle 2, 2, -1\rangle \cdot \langle 1, 2, 1\rangle}{\sqrt{2^2+2^2+(-1)^2}\cdot\sqrt{1^2+2^2+1^2}}$$

$$= \frac{2+4-1}{\sqrt{9}\cdot\sqrt{6}} = \frac{5}{3\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right) \approx$$

<u>Question 4</u> (a) $r = 1,\ \theta = \pi/2,\ z = 1$



$$x = r\cos\theta = \cos\frac{\pi}{2} = 0$$

$$y = r\sin\theta = \sin\frac{\pi}{2} = 1$$

the rectangular coordinates are $(0, 1, 1)$

(b) The spherical coordinates are (ρ, θ, ϕ)

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+3+4\times 3} = \sqrt{16} = 4$$

From $z = \rho\cos\phi$, we have $\cos\phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4}$

$\cos\phi = \frac{\sqrt{3}}{2}$, so $\phi = \pi/6$ or $30°$

From $x = \rho \sin\phi \cos\theta$, we have

$$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{1}{4\sin\frac{\pi}{6}} = \frac{1}{4\cdot\frac{1}{2}}$$

$$\cos\theta = \frac{1}{2} \qquad \theta = \frac{\pi}{3}$$

The spherical coordinates are $(4, \frac{\pi}{6}, \frac{\pi}{3})$

Question 5

(a) $y = r\sin\theta \qquad z = z$

$$r^2\sin^2\theta + z^2 = 1$$

(b) $y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$

$$\rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = 1$$

$$\rho^2(\sin^2\phi\sin^2\theta + \cos^2\phi) = 1$$

Question 6

(a) $\lim\limits_{t\to 0} \langle \frac{\sin t}{t}, e^{-t}, \frac{e^t - 1}{t} \rangle$

$$= \langle \lim_{t\to 0}\frac{\sin t}{t}, \lim_{t\to 0} e^{-t}, \lim_{t\to 0}\frac{e^t-1}{t} \rangle$$

$$\lim_{t\to 0} e^{-t} = e^{-0} = 1$$

Notice that $\lim\limits_{t\to a} \dfrac{f(t)-f(a)}{t\to a} = f'(a)$

If $f(t) = \sin t$ and $a = 0$, we find

$$\lim_{t\to 0} \frac{\sin t}{t} = \frac{\sin t - \sin 0}{t-0} = \cos 0 = 1$$

If $f(t) = e^t$, $a = 0$, we find

$$\lim_{t\to 0} \frac{e^t - 1}{t} = \frac{e^t - e^0}{t-0} = e^0 = 1$$

Therefore $\lim\limits_{t\to 0} \langle \frac{\sin t}{t}, e^{-t}, \frac{e^t - 1}{t} \rangle = \langle 1, 1, 1 \rangle$

(b) $r'(t) = \langle 2\cos t, -2\sin t, \sec^2 t \rangle$

$T(t) = \dfrac{r'(t)}{|r'(t)|}$, ~~$r(\frac{\pi}{4}) = \langle 2\sin\frac{\pi}{2}, 2\cos\frac{\pi}{4}, \tan\frac{\pi}{4} \rangle$~~

$r(\pi/4) = \langle \sqrt{2}, \sqrt{2}, 1 \rangle$

$$r'(\pi/4) = \langle 2\cos\frac{\pi}{4}, -\sin\frac{\pi}{4}, \sec^2\frac{\pi}{4} \rangle$$

$$= \langle \sqrt{2}, -\sqrt{2}, 2 \rangle$$

$$T(\pi/4) = \frac{r'(\pi/4)}{|r'(\pi/4)|} = \frac{\langle \sqrt{2}, -\sqrt{2}, 2 \rangle}{\sqrt{2+2+4}}$$

$$= \frac{\langle \sqrt{2}, -\sqrt{2}, 2 \rangle}{\sqrt{8}} = \frac{\langle \sqrt{2}, -\sqrt{2}, 2 \rangle}{2\sqrt{2}}$$

$$T(\pi/4) = \langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$$

(c) $$\int_0^1 \left(\frac{1}{1+t^2} i + \frac{1}{1+t} j + k \right) dt$$

$$= \left(\int_0^1 \frac{dt}{1+t^2} \right) i + \left(\int_0^1 \frac{dt}{1+t} \right) j + \left(\int_0^1 dt \right) k$$

$$\int_0^1 \frac{dt}{1+t^2} = \tan^{-1} t \Big]_0^1 = \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\int_0^1 \frac{dt}{1+t} = \ln|1+t| \Big]_0^1 = \ln(1+1) - \ln 1$$

$$= \ln 2$$

$$\int_0^1 dt = t \Big]_0^1 = 1 - 0 = 1$$

$$\int_0^1 \left(\frac{1}{1+t^2} i + \frac{1}{1+t} j + k \right) dt$$

$$= \frac{\pi}{4} i + \ln 2 \, j + k$$