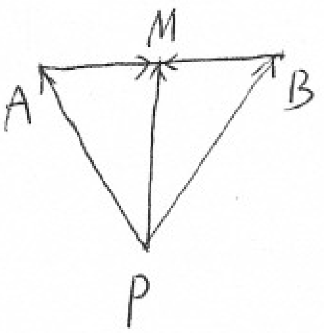
1.

a) Suppose that M is the midpoint of the segment AB in space and that P is another point. Show that $\overrightarrow{PM} = \frac{1}{2}(\overrightarrow{PA} + \overrightarrow{PB})$. **[4 marks]**

Solution.

$$\overrightarrow{PM} = \overrightarrow{PA} + \overrightarrow{AM} = \overrightarrow{PA} + \frac{1}{2}\overrightarrow{AB} \quad ①$$



$$\overrightarrow{PM} = \overrightarrow{PB} + \overrightarrow{BM} = \overrightarrow{PB} + \frac{1}{2}\overrightarrow{BA} \quad ②$$

$① + ② \Rightarrow$

$$2\overrightarrow{PM} = \overrightarrow{PA} + \frac{1}{2}\overrightarrow{AB} + \overrightarrow{PB} + \frac{1}{2}\overrightarrow{BA}$$

$$= \overrightarrow{PA} + \cancel{\frac{1}{2}\overrightarrow{AB}} + \overrightarrow{PB} - \cancel{\frac{1}{2}\overrightarrow{AB}} = \overrightarrow{PA} + \overrightarrow{PB}$$

$$\therefore \overrightarrow{PM} = \frac{1}{2}(\overrightarrow{PA} + \overrightarrow{PB})$$

b) Suppose that $\vec{a}$ and $\vec{b}$ are two unit vectors. Show that $\vec{a}+\vec{b}$ is perpendicular to $\vec{a}-\vec{b}$. **[4 marks]**

Solution.

$$\because (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$$

$$= \vec{a}\cdot(\vec{a}-\vec{b}) + \vec{b}\cdot(\vec{a}-\vec{b})$$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot(-\vec{b}) + \vec{b}\cdot\vec{a} + \vec{b}\cdot(-\vec{b})$$

$$= 1 - \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{b} - 1$$

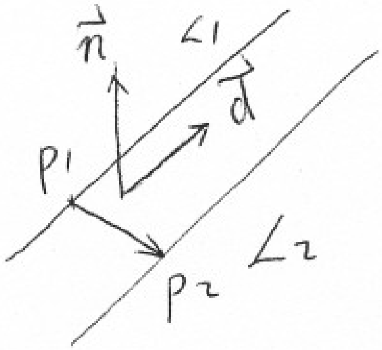
$$= 0$$

$$\therefore \vec{a}+\vec{b} \perp \vec{a}-\vec{b}$$

2. Find the equation of the plane that contains the two parallel lines

$$L_1: \begin{cases} x = 1 + t \\ y = -1 + 2t \\ z = 2 + 3t \end{cases} \text{ and } L_2: \begin{cases} x = 2 + 2s \\ y = 4s \\ z = 1 + 6s \end{cases}.$$

[6 marks]



Solution: $P_1(1, -1, 2)$ lies in L_1.

$P_2(2, 0, 1)$ lies in L_2, $\vec{d} = \langle 1, 2, 3 \rangle // L_1$.

$\overrightarrow{P_1P_2} = \langle 2-1, 0-(-1), 1-2 \rangle = \langle 1, 1, -1 \rangle$.

$\therefore$ $\vec{d}$ and $\overrightarrow{P_1P_2}$ are the two vectors parallel to the plane.

The normal vector $\vec{n}$ of the plane can be

$$\vec{n} = \overrightarrow{P_1P_2} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 5\vec{i} - 4\vec{j} + \vec{k}.$$

$\because$ P_1 lies in the plane.

$\therefore$ The equation of the plane is

$$5 \cdot (x-1) - 4 \cdot (y-(-1)) + 1 \cdot (z-2) = 0$$

or $\quad 5x - 4y + z - 11 = 0$.

3. Determine whether or not the four points $A(2,0,-3)$, $B(0,5,4)$, $C(1,1,-1)$ and $D(5,-12,-18)$ are coplanar. **[4 marks]**

Solution. $\overrightarrow{AB} = <0-2, 5-0, 4-(-3)> = <-2, 5, 7>$.

$\overrightarrow{AC} = <1-2, 1-0, -1-(-3)> = <-1, 1, 2>$.

$\overrightarrow{AD} = <5-2, -12-0, -18-(-3)> = <3, -12, -15>$.

$$\because \overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} -2 & 5 & 7 \\ -1 & 1 & 2 \\ 3 & -12 & -15 \end{vmatrix}$$

$$= -2\cdot 9 - 5\cdot 9 + 7\cdot 9 = 0$$

$\therefore$ The four points A, B, C and D are coplanar.

4. A particle moves along a curve C with velocity given by $\bar{v}(t) = <2t, e^t, \sin(\pi t)>$. At time $t=1$, the particle passes through the point $P(1,0,1/\pi)$. Determine the position vector $\bar{r}(t)$ at time t. **[4 marks]**

Solution. $\vec{r}(t) = \int \vec{v}(t)\,dt = \int <2t, e^t, \sin\pi t>\,dt$

$= <t^2, e^t, -\frac{1}{\pi}\cos\pi t> + \vec{C}$.

$\because \vec{r}(1) = \overrightarrow{OP} = <1, 0, \frac{1}{\pi}>$

$\therefore \vec{r}(1) = <1^2, e^1, -\frac{1}{\pi}\cos(\pi\cdot 1)> + \vec{C}$

$= <1, e, \frac{1}{\pi}> + \vec{C} = <1, 0, \frac{1}{\pi}>$

$\therefore \vec{C} = <1, 0, \frac{1}{\pi}> - <1, e, \frac{1}{\pi}> = <0, -e, 0>$

$\therefore \vec{r}(t) = <t^2, e^t, -\frac{1}{\pi}\cos\pi t> + <0, -e, 0>$

$= <t^2, e^t - e, -\frac{1}{\pi}\cos\pi t>$

5. A space curve is given by the vector function $\vec{r}(t) = <4t, t^2, 2t^2>$.

a) Find the equation of the line that is tangent to its position vector and that is parallel to the line: $\frac{x-1}{2} = y = \frac{z+1}{2}$. **[6 marks]**

Solution. $\vec{r}'(t) = <4, 2t, 4t>$.

$\because$ The tangent line // the line $\frac{x-1}{2} = y = \frac{z+1}{2}$.

$\therefore$ $\vec{r}'(t) = <4, 2t, 4t> // <2, 1, 2>$.

So the tangent line is the line through the point where $t = 1$.

$\vec{r}(1) = <4, 1, 2>$.

So, the equation of the tangent line is

$x = 4 + 4t,\ y = 1 + 2t,\ z = 2 + 4t,\ t \in R$.

b) Find the curvature of curve κ for $t \geq 0$.

(Hint: $\kappa = \left|\frac{d\bar{T}}{ds}\right| = \frac{1}{v}\left|\frac{d\bar{T}}{dt}\right| = \frac{|\bar{r}'(t) \times \bar{r}''(t)|}{|\bar{r}'(t)|^3}$). **[4 marks]**

Solution. $\vec{r}'(t) = <4, 2t, 4t>$, $\vec{r}''(t) = <0, 2, 4>$.

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2t & 4t \\ 0 & 2 & 4 \end{vmatrix} = -16\vec{j} + 8\vec{k}.$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{0^2 + (-16)^2 + 8^2} = 8\cdot\sqrt{5}.$$

$$|\vec{r}'(t)| = \sqrt{4^2 + (2t)^2 + (4t)^2} = \sqrt{16 + 20t^2}.$$

$$K = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{8\cdot\sqrt{5}}{(16 + 20t^2)^{3/2}}.$$

6.

a) Convert the equation $x^2 + y^2 = 4$ both to cylindrical and to spherical coordinates. **[4 marks]**

Solution. In Cylindrical coordinates,

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$

So, $(r\cos\theta)^2 + (r\sin\theta)^2 = 4 \Rightarrow r = 2.$

In spherical coordinates,

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$

So, $(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = 4$

$\Rightarrow \rho^2\sin^2\phi = 4, \ \therefore \rho = \dfrac{2}{\sin\phi}, \quad 0 < \phi < \pi$

b) Find the domain of the function $z = \sqrt{25 - x^2 - y^2}$. Sketch its level curves at $z = 0,3,4$ in the xy-plane. **[4 marks]**

Solution. Domain: $\{(x,y);\ 25 - x^2 - y^2 \geq 0\}$

$= \{(x,y);\ x^2 + y^2 \leq 25\}.$

Let $z = k$, $\sqrt{25 - x^2 - y^2} = k$, or $x^2 + y^2 = 25 - k^2$,

$k = 0, \quad x^2 + y^2 = 25.$

$k = 3, \quad x^2 + y^2 = 25 - 9 = 16,$

$k = 4, \quad x^2 + y^2 = 25 - 16 = 9.$