MATH 251
Simon Fraser University
Department of Mathematics
Burnaby Campus
Midterm Exam No. 1
Feb. 9, 2005
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DO NOT LIFT UP THE COVER PAGE UNTIL INSTRUCTED. Once the test begins, please check that all pages are intact. Answer all five questions, clearly showing your working. Each question is worth 8 marks. If you run out of space use the extra space on the back of each page, CLEARLY INDICATING WHERE THE SOLUTION CONTINUES AND THE NUMBER OF THE QUESTION THAT YOU ARE ANSWERING. Ordinary scientific calculators are allowed. The duration of the test is 40 minutes.

Name:

Question 1:

Question 2:

Question 3:

Question 4:

Question 5:

TOTAL:

1a) (4 marks) Consider the plane surface $x - 2y + z = 3$. Find equations for the intersection of this surface with each of the coordinate planes.

Answer. The coordinate planes are xy, xz and yz.

1. xy-plane. In this case $z = 0$ so the equation of the intersection of the surface with this plane is the *line* $x - 2y = 3$.

2. xz-plane. In this case $y = 0$ so the equation of the intersection of the surface with this plane is the *line* $x + z = 3$.

3. yz-plane. In this case $x = 0$ so the equation of the intersection of the surface with this plane is the *line* $z - 2y = 3$.

1b) (4 marks) Consider the **surface of revolution** generated by revolving the curve $y = x^2$ around the y-axis. By examining the evolution of a point $P(x_0, y_0)$ on the curve, determine the equation for the surface of revolution.

Answer. As the surface is generated, P traces a *circle* with radius x_0. This circle lies in the plane $y = y_0$ and is a contour curve. The equation of the circle is the level curve which is the projection of this contour curve onto the xz-plane. It's equation is:

$$x^2 + z^2 = x_0^2.$$

From the fact that $y = x^2$ on the original curve, we have $x_0^2 = y_0$. Therefore:

$$x^2 + z^2 = y_0.$$

But this is true for any point $P(x_0, y_0)$ on the curve. We can therefore replace y_0 with y to obtain the equation for the surface of revolution:

$$y = x^2 + z^2.$$

2a) (1 mark) Consider the curve C given by the equation $y = 2x^2$. Find a parametric representation of C and hence write down the position vector that generates C.

Answer. Let $x = t$. Then $y = 2x^2 = 2t^2$. Therefore the parametric representation is $x(t) = t$, $y(t) = 2t^2$.

The position vector is $r(t) = \vec{i}t + 2\vec{j}t^2$.

2b) (2 marks) Evaluate the unit tangent vector to C at the point $(x, y) = (1, 2)$.

Answer. The point $(1, 2)$ corresponds to $t = 1$. The unit tangent vector is given by the formula $\vec{T}(t) = \vec{v}(t)/v(t)$, where $\vec{v}$ is the velocity vector and v is the speed. We first evaluate $\vec{v}$ and v:

$$\vec{v}(t) = \vec{r}'(t) = \vec{i} + 4\vec{j}t, \quad v(t) = |\vec{v}(t)| = \sqrt{1 + 16t^2}.$$

Therefore:

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{\vec{i} + 4\vec{j}t}{\sqrt{1 + 16t^2}}.$$

At $t = 1$:

$$\vec{T} = \frac{1}{\sqrt{17}}(\vec{i} + 4\vec{j}).$$

2c) (3 marks) Evaluate the principle unit normal to C at the point $(x, y) = (1, 2)$.

Answer. The principal unit normal is given by the formula $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$. We first find $\vec{T}'(t)$:

$$\vec{T}'(t) = \vec{i}\frac{d}{dt}\left(\frac{1}{\sqrt{1+16t^2}}\right) + 4\vec{j}\frac{d}{dt}\left(\frac{t}{\sqrt{1+16t^2}}\right) = \frac{-16\vec{i}t + 4\vec{j}}{(1+16t^2)^{3/2}}$$

At $t = 1$ we then have:

$$\vec{T}' = \frac{1}{(17)^{3/2}}(-16\vec{i} + 4\vec{j}), \qquad |T| = \frac{\sqrt{272}}{17^{3/2}} = \frac{4\sqrt{17}}{17^{3/2}} = \frac{4}{17}.$$

Therefore, at $t = 1$:

$$\vec{N} = \frac{\vec{T}'(1)}{|\vec{T}'(1)|} = \left(\frac{-16\vec{i} + 4\vec{j}}{17^{3/2}}\right)\frac{17}{4} = \frac{1}{\sqrt{17}}(-4\vec{i} + \vec{j}).$$

Alternative solution. We use the orthogonality of $\vec{N}$ and $\vec{T}$, *i.e.* $\vec{T} \cdot \vec{N} = 0$. First we have to decide the direction of $\vec{N}$. At the point $(1, 2)$ on the graph $y = 2x^2$, the curve is bending upwards as we increase t. Therefore $\vec{N}$ has to point in a North-West direction (with North as the positive y direction). So, the $\vec{i}$ component of $\vec{N}$ is negative and the $\vec{j}$ component is positive.
Let $\vec{N} = -a\vec{i} = b\vec{j}$ where a and b are positive constants. Then, for orthogonality to $\vec{T}$, we must have $-a + 4b = 0$. Therefore $a = 4b$. Without loss of generality choose $b = 1$, giving $\vec{N} = k(-4\vec{i} + \vec{j})$ where k is a positive constant. To find k we use the fact that $\vec{N}$ is a unit vector. Therefore $|\vec{N}| = k\sqrt{17} = 1$. This gives $k = 1/\sqrt{17}$.

2d) (2 marks) Evaluate the curvature of C at the point $(x, y) = (1, 2)$.

Answer. Curvature is given by the formula:

$$\kappa = \frac{|x'y'' - x''y'|}{v^3}$$

Using $v(1) = \sqrt{17}$, $x'(1) = 1$, $x''(1) = 0$, $y'(1) = 4$, $y''(1) = 4$, $\kappa = 4/17^{3/2}$.

3a) Consider the vectors $\vec{u} = 2i - j + 4k$, $\vec{v} = i - 3j - 2k$ and $\vec{w} = i + 6j + k$.

(i) (1 mark) Are $\vec{u}$ and $\vec{w}$ perpendicular or parallel? Explain your answer.

Answer. A computation gives $\vec{u} \cdot \vec{w} = 0$. Therefore the vectors are perpendicular due to the relationship

$$\cos\theta = \frac{\vec{u} \cdot \vec{w}}{|\vec{u}||\vec{w}|}$$

where θ is the angle between them.

(ii) (3 marks) Find **two** vectors which are perpendicular to **both** $\vec{u}$ and $\vec{v}$.

Answer. The vectors $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ are perpendicular to both $\vec{u}$ and $\vec{v}$. We only need to find one of them because of the relationship $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. Let's find $\vec{u} \times \vec{v}$:

$$(2, -1, 4) \times (1, -3, -2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 1 & -3 & -2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 4 \\ -3 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix}$$

$$= 14\vec{i} + 8\vec{j} - 5\vec{k}$$

The two vectors are therefore $14\vec{i} + 8\vec{j} - 5\vec{k}$ and $-14\vec{i} - 8\vec{j} + 5\vec{k}$.

3b) (4 marks) Given the vectors $\vec{v} = (v_1, v_2, v_3)$, $\vec{u} = (u_1, u_2, u_3)$ and $\vec{w} = (w_1, w_2, w_3)$ show that:

$$\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$$

Answer.

$$\begin{aligned}
\vec{v} \cdot (\vec{u} + \vec{w}) &= (v_1, v_2, v_3) \cdot ((u_1, u_2, u_3) + (w_1, w_2, w_3)) \\
&= (v_1, v_2, v_3) \cdot (u_1 + w_1, u_2 + w_2, u_3 + w_3) \\
&= v_1(u_1 + w_1) + v_2(u_2 + w_2) + v_3(u_3 + w_3) \\
&= (v_1u_1 + v_2u_2 + v_3u_3) + (v_1w_1 + v_2w_2 + v_3w_3) \\
&= \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}
\end{aligned}$$

4a) (4 marks) Find both the cylindrical and spherical coordinates of the point $(0, 0, -1)$, given in rectangular Cartesian coordinates. **Note:** Rectangular coordinates are (x, y, z), cylindrical are (r, θ, z) and spherical are (ρ, ϕ, θ).

Answer.

1. Cylindrical. The z coordinate stays the same. The radial coordinate is given by:

$$r = \sqrt{x^2 + y^2} = 0$$

   The angle θ made with the positive x axis in a counter-clockwise direction can be taken to be 0 since both x and y are zero. Therefore the cylindrical coordinates are $(0, 0, -1)$.

2. Spherical. The radial distance is given by:

$$\rho = \sqrt{x^2 + y^2 + z^2} = 1$$

   The angle θ as defined above can again taken to be zero. The angle ϕ that P makes with the positive z axis is π. Therefore the spherical coordinates are $(1, \pi, 0)$.

4b) (4 marks) Convert the equation $2x + y - z = 4$ into both cylindrical and spherical coordinates.

Answer.

1. Cylindrical. Use $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$:

$$2r\cos\theta + r\sin\theta - z = 4$$

2. Spherical. Use $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$:

$$\rho(2\sin\phi\cos\theta + \sin\phi\sin\theta - \cos\phi) = 4$$

5a) (3 marks) Given the function $f(x,y) = \sqrt{1 - x^2 - 2y^2}$ determine its **largest possible domain**. Sketch the domain in the xy plane.

Answer. The largest possible domain is the region for which the argument of the square root is non-negative. It is given by $x^2 + 2y^2 \leq 1$.

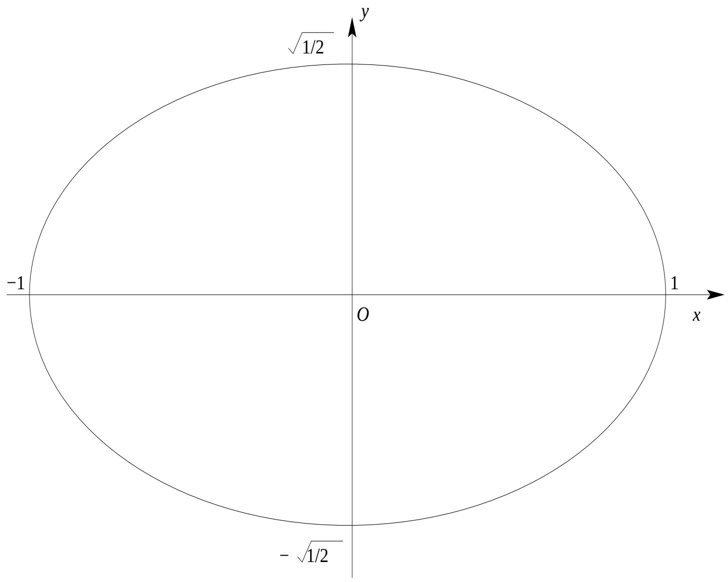

Figure 1: largest possible domain of $f(x,y) = \sqrt{1 - x^2 - 2y^2}$.

5b) Let $f(x, y) = x^2 + 2y^2$

(i) (3 marks) Describe and sketch some typical level curves of $f(x, y)$.

Answer. The level curves are given by projections onto the xy plane of the contour curves of the graph $z = f(x, y)$. Since $f(x, y)$ is positive, these contour curves only exist for $z \geq 0$. A typical level curve, for $z = R$, is $x^2 + 2y^2 = R$ which is the equation for an ellipse centred at the origin of the xy plane.

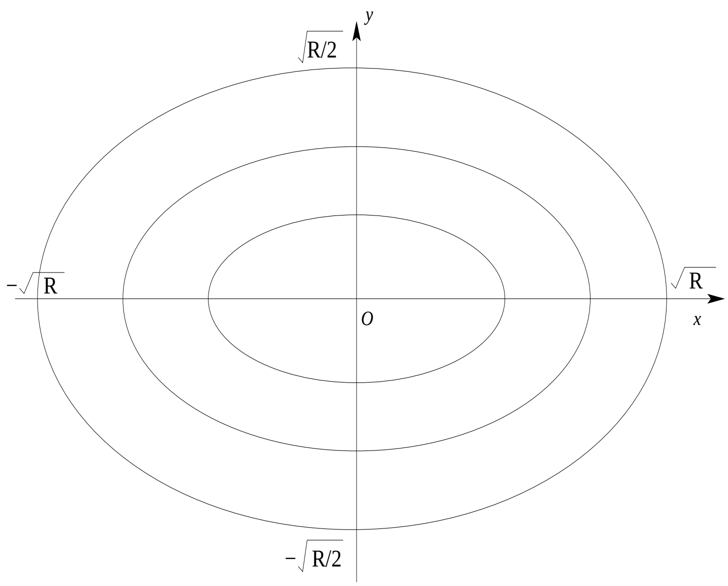

Figure 2: Typical level curves of $f(x, y) = x^2 + 2y^2$.

(ii) (2 marks) Using your previous answer sketch the graph of $f(x, y)$, indicating some typical contour curves.