MATH 251-3, Fall 2006 **Simon Fraser University**

Midterm 1 4 October 2006, 8:30–9:20am

Instructor: Ralf Wittenberg

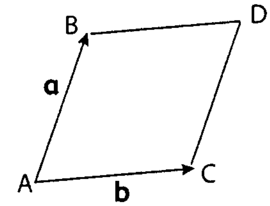
Last Name:	
First Name:	
SFU ID:	
Signature:	Answer key

INSTRUCTIONS

1. PLEASE DO NOT OPEN THIS BOOKLET UNTIL INVITED TO DO SO.
2. Write your last name, first name(s) and SFU ID in the box above in block letters, and sign your name in the space provided.
3. This exam contains 6 questions on 5 pages (after this title page). Once the exam begins please check to make sure your exam is complete.
4. The total time available is 50 minutes, and there are 50 points, so allow about a minute per point; for example, you should aim to spend about 10 minutes on a 10-point question. Attempt all problems!
5. This is a closed book exam. Only non-programmable scientific calculators are allowed.
6. Use the reverse side of the previous page if you need more room for your answer, and clearly indicate where the solution continues.
7. Show all your work, and explain your answers clearly.
8. Good luck!

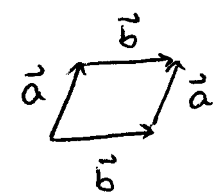
Question	Maximum	Score
1	10	
2	12	
3	8	
4	8	
5	8	
6	4	
Total	50	

1. Let $ABCD$ be a parallelogram as shown; that is, the sides satisfy $\overrightarrow{AB}=\overrightarrow{CD}$ and $\overrightarrow{AC}=\overrightarrow{BD}$.
Define $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{AC}$.



(a) [5 points]
Write expressions for the diagonals $\overrightarrow{AD}$ and $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Now assume that the lengths of the sides are equal, $|\mathbf{a}| = |\mathbf{b}|$: show that the diagonals $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are perpendicular.



$\vec{AD} = \vec{a} + \vec{b}$ $\qquad \vec{BC} = \vec{b} - \vec{a}$

$$\vec{AD} \cdot \vec{BC} = (\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a})$$
$$= \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} \qquad (\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$$
$$= |\vec{b}|^2 - |\vec{a}|^2$$
$$= 0 \quad \text{since } |\vec{a}| = |\vec{b}|$$

$\Rightarrow$ $\vec{AD}$, $\vec{BC}$ are perpendicular

(b) [5 points]
Now let the four points be given as

$$A(1,2,0),\ B(3,1,-2),\ C(1,2,3) \text{ and } D(3,1,1):$$

Verify that these points determine a parallelogram with equal side lengths, and find the area of the parallelogram $ABCD$.

$\vec{a} = \vec{AB} = \langle 2,-1,-2\rangle$, $\vec{b} = \vec{AC} = \langle 0,0,3\rangle$,
$\vec{BD} = \langle 0,0,+3\rangle$, $\vec{CD} = \langle 2,-1,-2\rangle$ $\Big\} \Rightarrow \vec{AB} = \vec{CD}$, $\vec{AC} = \vec{BD}$

$|\vec{AB}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$
$|\vec{AC}| = \sqrt{0^2 + 0^2 + 3^2} = \sqrt{9} = 3$ $\Big\}$ sides have equal length.

Area of ABCD is $|\vec{AB} \times \vec{AC}| = |\vec{a} \times \vec{b}|$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & -2 \\ 0 & 0 & 3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & -2 \\ 0 & 3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & -2 \\ 0 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = -3\hat{\imath} - 6\hat{\jmath} + 0\hat{k} = \langle -3, -6, 0\rangle$$

Area $= |\vec{a}\times\vec{b}| = \sqrt{(-3)^2 + (-6)^2} = \sqrt{9+36} = \sqrt{45}$

2. Consider the two planes

$$x - y + z = -3 \quad \text{and} \quad 2x - z = 4.$$

(a) [3 points]
Find the angle θ between the planes.

Normals to planes: $\vec{n}_1 = \langle 1, -1, 1 \rangle$ $\quad |\vec{n}_1| = \sqrt{1+1+1} = \sqrt{3}$

$\vec{n}_2 = \langle 2, 0, -1 \rangle$ $\quad |\vec{n}_2| = \sqrt{4+0+1} = \sqrt{5}$

Angle θ is given by $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{\langle 1,-1,1\rangle \cdot \langle 2,0,-1\rangle}{\sqrt{3}\,\sqrt{5}}$

$= \dfrac{2+0-1}{\sqrt{15}} = \dfrac{1}{\sqrt{15}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{15}}\right)$

(b) [6 points]
Find the parametric equations and symmetric equations for the line of intersection of the planes.

Method 1: The line of intersection ℓ lies in both planes, so it is perpendicular to both normals $\vec{n}_1, \vec{n}_2$.

Method 2 is: find 2 points on line

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

$$= 1\hat{\imath} + 3\hat{\jmath} + 2\hat{k} = \langle 1, 3, 2 \rangle$$

So the vector $\vec{v} = \langle 1,3,2 \rangle$ is parallel to the line.

Point on line: set $z = 0 \Rightarrow$ $2x - 0 = 4 \Rightarrow x = 2$

$x - y + 0 = -3 \Rightarrow y = x + 3 = 5$

So $(2, 5, 0)$ is on the line.

Equation of line (parametric): $\boxed{\vec{r}(t) = \langle 2,5,0 \rangle + t\langle 1,3,2 \rangle}$

$\underbrace{x = 2+t, \quad y = 5+3t, \quad z = 2t}_{\text{parametric form}}$ $\Rightarrow t = \boxed{\dfrac{x-2}{1} = \dfrac{y-5}{3} = \dfrac{z}{2}}$ symmetric equations

(c) [3 points]
Find the equation of the plane which passes through the point $P(1, 0, -2)$ and is perpendicular to the line of intersection found in (b).

Normal to plane is $\vec{v} = \langle 1,3,2 \rangle$ found in (b)

So the plane is $1(x-1) + 3(y-0) + 2(z-(-2)) = 0$

$\Rightarrow x - 1 + 3y + 2(z+2) = 0$

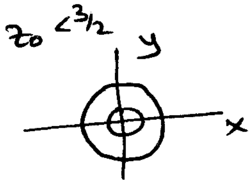
$\Rightarrow x + 3y + 2z = 1 - 4 = -3$

3. (a) [5 points]
Consider the surface described by the equation

$$x^2 + y^2 + 2z = 3.$$

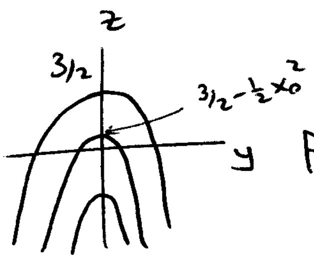
Briefly describe the traces (cross-sections) of the surface in horizontal planes of the form $z = z_0$, *and* the traces (cross-sections) of the surface in vertical planes of the form $x = x_0$. (You may wish to include a rough sketch in your description.) Can you identify the type of surface?

Horizontal traces: $z = z_0 \Rightarrow x^2+y^2+2z_0 = 3$
$\Rightarrow x^2+y^2 = 3-2z_0$



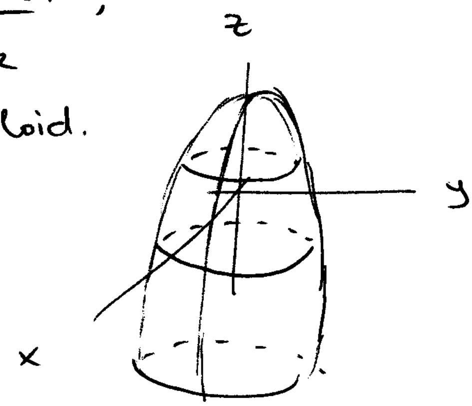
for $z_0 > 3/2$, no solutions
for $z_0 \le 3/2$, circles of radius $\sqrt{3-2z_0}$

Vertical traces: $x = x_0 \Rightarrow x_0^2+y^2+2z = 3$
$\Rightarrow z = \frac{3}{2} - \frac{1}{2}y^2 - \frac{1}{2}x_0^2$
$= \left(\frac{3}{2} - \frac{1}{2}x_0^2\right) - \frac{1}{2}y^2$



Parabolas (parallel to) in y-z plane, opening down, maximum value $3/2 - \frac{1}{2}x_0^2 \le 3/2$.

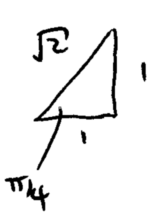
The surface is a (circular) paraboloid.



(b) [3 points]
Consider the point P on the surface given in rectangular (Cartesian) coordinates as $(x, y, z) = (0, -1, 1)$. Rewrite the point P in spherical coordinates (ρ, θ, ϕ).

$$\left.\begin{aligned} x &= \rho\sin\phi\cos\theta = 0 \\ y &= \rho\sin\phi\sin\theta = -1 \\ z &= \rho\cos\phi = 1 \end{aligned}\right\} \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+1} = \sqrt{2}$$

$\rho\cos\phi = 1 \Rightarrow \cos\phi = \frac{1}{\sqrt{2}}$, $0 \le \phi \le \pi$
$\Rightarrow \phi = \pi/4$



$\rho\sin\phi\sin\theta = -1 \Rightarrow \sin\theta = \frac{-1}{\sqrt{2}\cdot 1/\sqrt{2}} = -1$
$\Rightarrow \theta = 3\pi/2$

Coordinates:
$(\rho, \theta, \phi) = (\sqrt{2}, 3\pi/2, \pi/4)$

4. (a) [4 points]
Find a vector function that parametrizes the curve C of intersection of the paraboloid $4x^2 + z^2 = y$ and the parabolic cylinder $z = 2 - x^2$.

Use $x = t$ as parameter: $x = t$

$z = 2 - x^2 = 2 - t^2$

$y = 4x^2 + z^2 = 4t^2 + (2-t^2)^2$

$= 4t^2 + 4 - 4t^2 + t^4 = 4 + t^4$

$\Rightarrow \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$= \langle t, 4 + t^4, 2 - t^2 \rangle$

(b) [2 points]
Find the intersection point P of the curve C found in (a) with the plane $x = 1$.

When $x = 1$, we have $t = 1$.

$\vec{r}(1) = \langle 1, 5, 1 \rangle$ so P is the point $(1, 5, 1)$

(c) [2 points]
Compute the tangent vector to the curve C at the point P.

Tangent to curve $\vec{r}\,'(t) = \langle 1, 4t^3, -2t \rangle$

At P, we have $t = 1$, so the tangent at P is $\vec{r}\,'(1) = \langle 1, 4, -2 \rangle$

5. A particle moves up along a helical wire which extends from $z = 0$ to $z = 8$; the position of the particle is given as a function of time t by

$$\mathbf{r}(t) = \left\langle 8\sin\left(\frac{t}{2}\right), -8\cos\left(\frac{t}{2}\right), 3t \right\rangle.$$

   (a) [5 points]
   Find the velocity vector $\mathbf{v}(t)$ and acceleration vector $\mathbf{a}(t)$, and the speed of the particle $v(t)$, as a function of t.
   Show that the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ are always orthogonal.

Velocity $\vec{v}(t) = \vec{r}'(t) = \langle 4\cos\frac{t}{2}, 4\sin t/2, 3\rangle$

Acceleration $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle -2\sin t/2, 2\cos t/2, 0\rangle$

Speed $v = |\vec{v}| = \sqrt{4^2\cos^2 t/2 + 4^2\sin^2 t/2 + 3^2}$

$= \sqrt{16(\cos^2 t/2 + \sin^2 t/2) + 9} = \sqrt{16+9} = \sqrt{25} = 5$

$\vec{v}(t)\cdot\vec{a}(t) = -8\cos t/2 \sin t/2 + 8\sin t/2 \cos t/2 + 0 = 0 \Rightarrow \vec{v} \perp \vec{a}$

   (b) [3 points]
   Determine the time at which the particle reaches the upper end of the wire, and compute the total distance travelled by the particle (that is, the length of the wire).

Upper end of wire is at $z = 8$

So $3t = 8 \Rightarrow t = 8/3$.

Length $L = \int_0^{8/3} |\vec{r}'(t)|\,dt = \int_0^{8/3} v(t)\,dt = \int_0^{8/3} 5\,dt$

$= 5t\Big|_0^{8/3} = 5\cdot\frac{8}{3} = \frac{40}{3}$

6. [4 points]
   Let $\mathbf{u} = \mathbf{r}\cdot(\mathbf{r}' \times \mathbf{r}'')$
   (where $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}''$, $\mathbf{u}$ are all functions of t, so that $\mathbf{r} = \mathbf{r}(t)$, $\mathbf{u} = \mathbf{u}(t)$ etc.).
   Differentiate the expression for $\mathbf{u}$ carefully to show that

$$\mathbf{u}' = \mathbf{r}\cdot(\mathbf{r}' \times \mathbf{r}'''),$$

   and explain your reasoning.

$\vec{u} = \vec{r}\cdot(\vec{r}' \times \vec{r}'') \Rightarrow \vec{u}' = \vec{r}'\cdot(\vec{r}' \times \vec{r}'') + \vec{r}\cdot\frac{d}{dt}(\vec{r}' \times \vec{r}'')$

(product rule for dot product)

$\Rightarrow \vec{u}' = \underbrace{\vec{r}'\cdot(\vec{r}' \times \vec{r}'')}_{=0} + \vec{r}\cdot(\underbrace{\vec{r}'' \times \vec{r}''}_{=\vec{0}}) + \vec{r}\cdot(\vec{r}' \times \vec{r}''')$

(product rule for cross product)

$\vec{r}'' \times \vec{r}'' = \vec{0}$ (cross product of vector with itself)

$\vec{r}' \times \vec{r}'' \perp \vec{r}' \Rightarrow \vec{r}'\cdot(\vec{r}' \times \vec{r}'') = 0$ So $\vec{u}' = \vec{r}\cdot(\vec{r}' \times \vec{r}''')$