[6] 1. Calculate the length of the curve parametrized by

$$\mathrm{r}(t) = \langle t, \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}, \frac{t^2}{2}\rangle$$

from $t = 0$ to $t = 2$.

$$s = \int_0^2 v(t)\,dt = \int_0^2 \sqrt{r_x(t)^2 + r_y(t)^2 + r_z(t)^2}\,dt$$

$$= \int_0^2 \sqrt{1^2 + \left[\frac{2\sqrt{2}}{3}\cdot\frac{3}{2}t^{\frac{1}{2}}\right]^2 + \left[\frac{1}{2}\cdot 2t\right]^2}\,dt$$

$$= \int_0^2 \sqrt{1 + 2t + t^2}\,dt = \int_0^2 (1+t)\,dt = \left[t + \frac{t^2}{2}\right]_0^2 = 4$$

[4] 2. Find the equation of the surface generated by revolving the curve $y = e^{-z}$ around the z-axis.

curve: $f(y, z) = y - e^{-z} = 0$

surface: $f(\sqrt{x^2+y^2}, z) = \sqrt{x^2+y^2} - e^{-z} = 0$

$$\Rightarrow \sqrt{x^2+y^2} = e^{-z}$$

$$\Rightarrow x^2 + y^2 = e^{-2z}$$

use $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

[8] 3. Let $\mu(t) = \mathbf{r}(t) \times \mathbf{v}(t)$, and $\tau(t) = \mathbf{r}(t) \times \mathbf{a}(t)$, where $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ are the position, velocity, and acceleration of a particle moving through space as functions of time. Prove that

$$\mu'(t) = \tau(t).$$

$$\mu(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \end{vmatrix} = \begin{bmatrix} y(t)z'(t) - z(t)y'(t) \\ z(t)x'(t) - x(t)z'(t) \\ x(t)y'(t) - y(t)x'(t) \end{bmatrix}$$

$$\mu'(t) = \begin{bmatrix} y'(t)z'(t) + y(t)z''(t) - [z(t)y''(t) + z'(t)y'(t)] \\ \text{etc.} \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} y(t)z''(t) - z(t)y''(t) \\ z(t)x''(t) - x(t)z''(t) \\ x(t)y''(t) - y(t)x''(t) \end{bmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x(t) & y(t) & z(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \vec{r}(t) \times \vec{a}(t) = \tau(t)$$

$\square$.

[8] 4. Using the definition of curvature and a general parametric equation of a line in three dimensions, prove that the curvature of a straight line in space is zero.

$$K = \left| \frac{d\phi}{ds} \right| = \left| \frac{d\vec{T}}{ds} \right|$$

where $\vec{T} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$ where $\vec{v}(t) = \vec{r}\,'(t)$

Straight line :

$\vec{r}(t) = \vec{r}_0 + t\vec{v}_0$ $\vec{r}_0, \vec{v}_0$ are constant vectors

$\therefore \vec{r}\,'(t) = \vec{v}_0$, constant

$\Rightarrow \vec{T} = \dfrac{\vec{v}_0}{|\vec{v}_0|}$, constant

$$\therefore \left| \frac{d\vec{T}}{ds} \right| = 0 = K$$

$\square$

see →

[6] 5. Given $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 + z^2}$,
prove, using spherical coordinates, whether or not

$$\lim_{(x,y,z)\to(0,0,0)} f(x, y, z)$$

exists. If the limit exists, find it.

$$f(x,y,z) = \frac{(\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\theta)}{\rho^2}$$

$$= \frac{\rho^3 g(\phi, \theta)}{\rho^2} \qquad \text{where} \quad -1 \leq g(\phi,\theta) \leq 1$$

$$= \rho\, g(\phi, \theta) \to 0 \quad \text{as} \quad \rho \to 0$$

$\therefore$ the limit exists, and

$$\lim_{(x,y,z)\to(\vec{0})} f(x,y,z) = 0$$

[5] 6. Use a double integral to find the volume bounded above by the paraboloid $x^2 + y^2 + \frac{z}{2} = 1$ and below by the xy-plane.

polar coords:

$z = 2 - 2r^2$

$z = 0 = 2 - 2r^2$
$\Rightarrow r^2 = 1 \Rightarrow r = 1$

$$V = \int_0^{2\pi} \int_0^1 (2 - 2r^2)\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^2}{4}\right]_0^1 d\theta$$

$$= 2\left[\frac{1}{2} - \frac{1}{4}\right] 2\pi$$

$$= \pi$$

[8] 7. Given $\phi(x,y,z) = \frac{1}{r}$, where $r = \sqrt{x^2+y^2+z^2}$, prove that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \ .$$

$$\frac{\partial \phi}{\partial x} = -x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

and $$\frac{\partial^2 \phi}{\partial x^2} = -\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} + 3x^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$$
$$= \left[-\left(x^2+y^2+z^2\right) + 3x^2\right]\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$$

similarly, $$\frac{\partial^2 \phi}{\partial y^2} = \left[-\left(x^2+y^2+z^2\right) + 3y^2\right]\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$$

and $$\frac{\partial^2 \phi}{\partial z^2} = \left[-\left(x^2+y^2+z^2\right) + 3z^2\right]\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \left[-3\left(x^2+y^2+z^2\right) + 3x^2 + 3y^2 + 3z^2\right]\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$$

$$= 0$$

$\square$

[8] 8. Find the maximum value of $f(x, y, z) = xyz$, on the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $z = 1/2$, using the method of Lagrange multipliers.

constraints: $g(x,y,z) = x^2+y^2+z^2-1 = 0$

$h(x,y,z) = z - \frac{1}{2} = 0 \Rightarrow z = \frac{1}{2}$

$f_x(x,y,z) = yz = \lambda g_x(x,y,z) + \mu h_x(x,y,z)$

$\Rightarrow yz = \lambda 2x + \mu 0$

& $xz = \lambda 2y + \mu 0$

$xy = \lambda 2z + \mu$

$\therefore \left.\begin{array}{l} xyz = \lambda 2x^2 \\ xyz = \lambda 2y^2 \end{array}\right\} \Rightarrow \begin{array}{l} x^2 = y^2 \\ x = \pm y \end{array}$

$xyz = \lambda 2z^2 + \mu z$

using $x^2+y^2+z^2-1=0$:

$x^2+x^2+\left(\frac{1}{2}\right)^2-1=0$

$\Rightarrow 2x^2 = \frac{3}{4}$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2\sqrt{2}} = \pm y$

Maximum value at $\left(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2}\right)$

or $\left(-\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2}\right)$

$f_{max} = \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2\sqrt{2}}\right)\frac{1}{2} = \frac{3}{16}$

[6] 9. Use triple integration to find the moment of inertia around the z-axis of the solid bounded above by $z = 3$ and below by the paraboloid $z = 2x^2 + 2y^2$. Assume density $\delta \equiv 1$.

and $2r^2 = 3 \Rightarrow r = \frac{\sqrt{3}}{\sqrt{2}}$

cylindric coords: $z = 2r^2$

$$I_z = \iiint (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\frac{\sqrt3}{\sqrt2}}\int_{2r^2}^{3} r^2\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\sqrt3}{\sqrt2}} (3-2r^2)\,r\,r^2\,dr\,d\theta = 2\pi\left[\frac{3r^4}{4} - \frac{2r^6}{6}\right]_0^{\frac{\sqrt3}{\sqrt2}}$$

$$= 2\pi\left[\frac{27}{16} - \frac{18}{16}\right] = \frac{9}{8}\pi$$

[6] 10. Find the surface area of the surface parametrized by $\mathbf{r}(u, v) = \langle \sin u, \cos u, v\rangle$ for $0 \le u \le 2\pi$ and $1 \le v \le 2$.

$$A = \iint_R \left|\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}\right| du\,dv$$

$$\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v} = \begin{vmatrix} \vec i & \vec j & \vec k \\ \cos u & -\sin u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle -\sin u, -\cos u, 0\rangle$$

so $\left|\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}\right| = \sqrt{\sin^2 u + \cos^2 u} = 1$

$$= \int_0^{2\pi}\int_1^2 1 \cdot dv\,du$$

$$= 2\pi$$

[8] 11. Use the transformation given by $x = 2r\cos\theta$ and $y = r\sin\theta$ to find the volume of the solid bounded by the xy-plane, the paraboloid $z = x^2 + y^2$, and the elliptic cylinder $\frac{x^2}{4} + y^2 = 1$. (Use the fact that $\int \cos^2\theta\, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$.)

$$J_T(r,\theta) = \begin{vmatrix} x_r(r,\theta) & x_\theta(u,\theta) \\ y_r(r,\theta) & y_\theta(r,\theta) \end{vmatrix} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= 2r\cos^2\theta + 2r\sin^2\theta = 2r$$

$$\therefore V = \iint_R z\, dx\, dy = \int_0^{2\pi}\int_0^1 (4r^2\cos^2\theta + r^2\sin^2\theta)\, 2r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{8r^4}{4}\cos^2\theta + \frac{2r^4}{4}\sin^2\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left[2\cos^2\theta + \frac{1}{2}\sin^2\theta\right] d\theta$$

$$\frac{1}{2}\sin^2\theta = \frac{1}{2}(1-\cos^2\theta)$$

$$= \int_0^{2\pi}\left[\frac{1}{2} + \frac{3}{2}\cos^2\theta\right] d\theta$$

$$= \pi + \frac{3}{2}\left[\frac{2\pi}{2} + 0\right] = \pi + \frac{3\pi}{2} = \frac{5\pi}{2}$$

[5] 12. Prove that curl(grad$f$) $= 0$, if the real-valued function $f(x, y, z)$ has continuous second-order partial derivatives.

$$\text{grad } f = \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\text{curl}(\text{grad } f) = \vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{bmatrix}$$

$$= \vec{0},$$

since contin 2^{nd}-order partials $\Rightarrow \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$ etc.

[6] 13. Use the vector form of Green's theorem to calculate the flux of the field $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$ across the circle $x^2 + y^2 = 4$.

$\vec{F} = \langle P, Q \rangle$

$\vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$

$$\Phi = \oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \vec{\nabla} \cdot \vec{F}\, dA$$

$$= \iint_R (2x + 2y)\, dx\, dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x + 2y)\, dy\, dx = \int_{-2}^{2} 4x\sqrt{4-x^2}\, dx$$

$$= \left[-\frac{4}{3} \sqrt{4-x^2}^{\,3} \right]_{-2}^{2} = 0$$

[8] 14. Find the centroid of a half-turn of wire of uniform density given by $\mathbf{r}(t) = \langle 5\cos t, 5sint, t\rangle$, letting t run from 0 to π. Show either that the centroid is on the wire or that it is not on the wire.

$$s = \int_0^{\pi} |r'(t)|\,dt = \int_0^{\pi} \sqrt{(-5\sin t)^2 + (5\cos t)^2 + 1^2}\,dt$$

$$= \sqrt{26}\,\pi$$

$$\therefore \quad \bar{x} = \frac{1}{\sqrt{26}\,\pi}\int_0^{\pi} 5\cos t\,\sqrt{26}\,dt = \frac{1}{\pi}5\left[\sin t\right]_0^{\pi} = 0$$

$$\bar{y} = \frac{1}{\sqrt{26}\,\pi}\int_0^{\pi} 5\sin t\,\sqrt{26}\,dt = \frac{1}{\pi}5\left[-\cos t\right]_0^{\pi} = \frac{10}{\pi}$$

$$\bar{z} = \frac{1}{\sqrt{26}\,\pi}\int_0^{\pi} t\sqrt{26}\,dt = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$\therefore$ Centroid is

$$\left(0, \frac{10}{\pi}, \frac{\pi}{2}\right)$$

Not on wire:

$t = \frac{\pi}{2}$

$\Rightarrow 5\sin t = 5$

$\neq \frac{10}{\pi}$

[8] 15. Show that the vector field $\mathbf{F}(x, y) = \langle y\cos x + \cos y, \sin x - x\sin y\rangle$ is conservative. Then find a potential function for $\mathbf{F}$ by the method of integration.

$$P = y\cos x + \cos y$$
$$Q = \sin x - x\sin y$$

$$\frac{\partial P}{\partial y} = \cos x - \sin y = \frac{\partial Q}{\partial x}$$

$\Rightarrow \vec{F}$ is conservative.

Want $f(x,y)$ s.t. $\vec{F} = \vec{\nabla} f$.

Integrating P:

$$\frac{\partial f}{\partial x} = y\cos x + \cos y$$

$$\Rightarrow f(x,y) = y\sin x + x\cos y + \alpha(y)$$

and $Q = \dfrac{\partial f}{\partial y} = \sin x - x\sin y + \alpha'(y)$

$$\Rightarrow \alpha'(y) = 0$$

$\therefore f(x,y) = y\sin x + x\cos y + C$

is a potential fn for $\vec{F}$.