MATH 251-3, Spring 2006 **Simon Fraser University**

Final Exam 13 April 2006, 7:00–10:00pm

Instructor: Ralf Wittenberg

Last Name:	______________________________
First Name:	______________________________
SFU ID:	______________________________
Signature:	______________________________

INSTRUCTIONS

1. PLEASE DO NOT OPEN THIS BOOKLET UNTIL INVITED TO DO SO.
2. Write your last name, first name(s) and student number in the box above in block letters, and sign your name in the space provided.
3. This exam contains 12 questions on 12 pages (after this title page). Once the exam begins please check to make sure your exam is complete.
4. The total time available is 3 hours (180 minutes), and there are 100 points; plan your time carefully. Attempt all problems!
5. This is a closed book exam. Only non-programmable scientific calculators are allowed.
6. Use the reverse side of the previous page if you need more room for your answer, and clearly indicate where the solution continues.
7. Show all your work (except for Question 1, where you only need to answer True or False), and explain your answers clearly.
8. Good luck!

Question	Maximum	Score
1	8	
2	6	
3	10	
4	8	
5	6	
6	8	
7	10	
8	12	
9	10	
10	10	
11	6	
12	6	
Total	100	

1. [8 points]
Answer True (T) or False (F) to the following questions; you do *not* need to show your working:

(a) The cross product of two unit vectors is a unit vector.

(b) The planes $3x - y - 4z = 0$ and $3x - 2y - 4z = 5$ intersect.

(c) If C is the curve with parametrization $x(t) = 4\cos 2t$, $y(t) = 4\sin 2t$, $0 \leq t \leq \pi$, then

$$\int_C 1\, ds = 4\pi.$$

(d) Suppose $f(x, y)$ approaches the limit L as (x, y) approaches (a, b) *along every straight line* through (a, b). Then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

(e) There exists a function $f(x, y)$ whose partial derivatives are $f_x = 3x + 2y^2$, $f_y = 3x - 2y^2$.

(f) Suppose the partial derivatives of the function $f(x, y)$ exist and are continuous. If $f(x, y)$ has a local minimum at the point $(-1, 4)$, then the equation of the tangent plane to the graph of the function f at the point $(-1, 4)$ is

$$z = f(-1, 4).$$

(g) The function $w(x, y) = x^2 + y^2$ satisfies the partial differential equation

$$xw_x + yw_y = 2w.$$

(h)

$$\int_0^1 \int_{y^3}^{\sqrt{y}} x^2 y\, dx\, dy = \int_0^1 \int_{x^{1/3}}^{x^2} x^2 y\, dy\, dx.$$

2. [6 points]

(a) Find the line of intersection of the planes

$$2x + y - 2z = 3 \qquad \text{and} \qquad x - 3y - z = -2.$$

(b) Find an equation for the plane that contains the origin and the line determined in part (a).

3. [10 points]

(a) Consider motion along a curve C parametrized by $\mathbf{r}(t)$. Recall that the acceleration vector $\mathbf{a} = \mathbf{r}''$ can be decomposed into its tangential and normal components as

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2 \mathbf{N},$$

where v is the speed, κ is the curvature, and $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and principal normal vectors, respectively. Use this equation to prove that

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{v^3},$$

where $\mathbf{v} = \mathbf{r}'$.

(b) A particle moves in the plane according to the parametric equations

$$x = 2\cos 2t, \qquad y = \sin 2t.$$

Compute the velocity and acceleration vectors and the speed, and use the formula in part (a) to find the curvature. Find the point(s) on the curve where the curvature is maximum.

4. [8 points]

(a) Derive a formula for the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (x_0, y_0, z_0), and show that it can be written in the form

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

(b) The equation

$$z\ln(z) = 4e^{x^2} - y^2$$

defines z implicitly in terms of the independent variables x and y. Find $\dfrac{\partial z}{\partial y}$ when $(x, y, z) = (0, -2, 1)$.

5. [6 points]
Suppose that the temperature (in degrees Celsius) at a point (x, y) in the plane at time t (in minutes) is given by the formula

$$T(x, y, t) = (3x + y^2)(1 + t).$$

Suppose an ant walks along a curve in the plane, so that its position at time t is $x(t) = 1 + 2t$, $y(t) = t^2$.
Using the chain rule, find the *rate of change of temperature with respect to time* experienced by the ant at time $t = 1$.

6. [8 points]
Let C be the curve of intersection of the two surfaces

$$x^2 + y^2 = 4 \qquad \text{and} \qquad 2x - y + z = 3.$$

Find the point(s) on C where the distance to the xz plane attains its maximum value. Clearly explain your reasoning.
[Use the method of Lagrange multipliers: Begin by writing down the function to be maximized and the constraints, and setting up an appropriate system of equations to be solved.]

7. [10 points]

   (a) Consider the double integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\, dx\, dy.$$

   Sketch the region over which the integration is being performed, and evaluate the integral.

   (b) Show that

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \frac{\pi}{4}.$$

8. [12 points]

(a) Compute the volume of the solid that lies above the paraboloid $z = x^2 + y^2$ and inside the ellipsoid $2x^2 + 2y^2 + z^2 = 24$.

(b) Find the area of the part of the surface

$$z = x^2 + y$$

that lies above the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$.

9. [10 points]
Consider the triple integral

$$\iiint_T z^2 \, dV,$$

where T is the region in space described in Cartesian (rectangular) coordinates by

$$x \geq 0, \qquad y \geq 0, \qquad z \geq 3, \qquad x^2 + y^2 + z^2 \leq 12.$$

Without attempting to evaluate the integral, write down the form of this triple integral, *explicitly indicating the limits of integration*, in

(a) Cartesian (rectangular) coordinates x, y, z.

(b) Cylindrical coordinates r, θ, z.

(c) Spherical coordinates ρ, ϕ, θ.

10. [10 points]

(a) Consider the region R in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$ and $xy = 3$. Sketch the region R in the xy plane, and the image of R in the uv plane under the transformation $x = u/v$, $y = v$. Find the Jacobian of the transformation.

(b) Evaluate the integral

$$\iint_R xy \, dx \, dy,$$

using the change of variables (transformation) described in (a).

11. [6 points]

(a) Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = e^{2z}\cos(x^2)\mathbf{i} + e^{2z}\sin(x^2)\mathbf{j} - 4xyz\mathbf{k}.$$

Calculate the divergence of $\mathbf{F}$.

(b) Assume that the real-valued function $f(x, y, z)$ has continuous second-order partial derivatives. Prove that

$$\text{curl}(\text{grad } f) = \mathbf{0}.$$

12. [6 points]
Let C be the twisted cubic curve parametrized by

$$x = t, \quad y = t^2, \quad z = t^3 \qquad 0 \leq t \leq 1.$$

Evaluate the following line integrals along the curve C:

(a)

$$\int_C xz\,dy.$$

(b)

$$\int_C \mathbf{F} \cdot \mathbf{T}\,ds,$$

where $\mathbf{F} = 4z\mathbf{i} + x\mathbf{j} - y\mathbf{k}$.