MATH 251
Simon Fraser University
Department of Mathematics
Burnaby Campus
Final Exam
Saturday April 16, 2005
Instructor: Dr. Akeel Shah

DO NOT LIFT UP THE COVER PAGE UNTIL INSTRUCTED. Once the test begins, please check that all pages are intact. Answer all ten questions, clearly showing your working. Each question is worth 10 marks. If you run out of space use the extra space on the back of each page, CLEARLY INDICATING WHERE THE SOLUTION CONTINUES AND THE NUMBER OF THE QUESTION THAT YOU ARE ANSWERING. Ordinary scientific calculators are allowed. The duration of the exam is 3 hours.

Name:

Question 1:

Question 2:

Question 3:

Question 4:

Question 5:

Question 6:

Question 7:

Question 8:

Question 9:

Question 10:

TOTAL:

1a) (4 marks) For vectors $\vec{a} = (5, 2, -1)$, $\vec{b} = (-1, 2, -4)$ and $\vec{c} = (-3, 0, 2)$, find

(i) $\vec{a} \times \vec{b}$.

(ii) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{b})$.

1b) (6 marks) Using the substitution $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\phi$, show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xz^2}{x^2 + y^2 + z^2} = 0.$$

2a) (6 marks) Let the acceleration vector of a particle in space be

$$\vec{a}(t) = 2e^t\vec{\imath} + 3t^2\vec{\jmath} + 4t^3\vec{k}$$

(i) If the velocity vector at $t = 0$ is $\vec{v}(0) = \vec{\imath} - \vec{\jmath}$, find the velocity vector at time t.

(ii) Find the speed at time $t = 2$.

2b) (4 marks) For two vector-valued functions

$$\vec{u}(t) = u_1(t)\vec{i} + u_2(t)\vec{j} \quad \text{and} \quad \vec{v}(t) = v_1(t)\vec{i} + v_2(t)\vec{j}$$

show that:

$$\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{v}'(t) \cdot \vec{u}(t)$$

where primes denote differentiation with respect to t.

3a) (6 marks) A particle moving in the plane has the position vector

$$\vec{r}(t) = (e^{2t} \cos t, e^{2t} \sin t)$$

Let the position vector generate a curve C. Find the curvature of C at time t.

3b) (4 marks) Find the tangential and normal components of acceleration of the particle.

4) (10 marks) Find the maximum and minimum values of the function

$$f(x, y) = 3x^3 + y^2 - 4x + y$$

on the triangular region R with vertices $(0, 0)$, $(0, 2)$ and $(2, 0)$.

5) (10 marks) Given the function

$$f(x, y, z) = x^3 + 2xy^2 + z$$

and the point $P(1, 2, 1)$

(i) Find the maximum directional derivative of f at P and the direction in which it occurs.

(ii) Find the directional derivative of f at P in the direction of $\vec{v} = (0, 3, 4)$.

(iii) Find an equation for the plane tangent to the surface $f(x, y, z) = 0$ at the point P.

6a) (5 marks) Find the area of the region R bounded by the curves $y = x^2$ and $y = 3 - x^2$.

6b) (5 marks) Evaluate the integral of the function $f(x, y) = xy$ over the region S bounded by the curve $y = x^2$ and the line $y = x$.

7) (10 marks) Consider the lamina that lies inside the limaçon $r = 1 + 2\cos\theta$ and outside the circle $r = 2$, and has density $\delta = r$. Find the mass of the lamina.

Hint: You will need the following integrals:

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx, \quad n = 1, 2, 3, \ldots$$

8) (10 marks) Evaluate the triple integral

$$\iiint_V (x+y)\, dV$$

where V is the region between the surfaces $z = 2 - x^2$ and $z = x^2$ for $0 \leq y \leq 3$.

9) (10 marks) Find the area of the first-quadrant region bounded by the curves

$$y = x^2,\ y = 2x^2 \quad \text{and} \quad x = y^2,\ x = 4y^2$$

by transforming from the coordinates (x, y) to (u, v), where

$$u = \frac{y}{x^2}, \quad v = \frac{x}{y^2}$$

10a) (5 marks) Find the divergence and curl of the vector field $\vec{F}(x, y, z) = (e^{x^2 y}, xe^y, ye^{z^2})$.

10b) (5 marks) Show that the vector field $\vec{F} = (y\cos(xy) + \sin(x), x\cos(xy) + \sin(y))$ is conservative and find its potential function.