MATH 251 Test 2 Solutions

Total Marks: 57.

(1) [5] Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$$\mathbf{r}(t) \;=\; 8\sin t\,\mathbf{i} \;+\; t\,\mathbf{j} \;+\; 8\cos t\,\mathbf{k}$$

$$\begin{aligned}
\text{arc length } = s(t) \;&=\; \int_0^t |\mathbf{r}'(t)|\,dt \\
&=\; \int_0^t \sqrt{(8\cos t)^2 + (1)^2 + (8\sin t)^2}\,dt \\
&=\; \int_0^t \sqrt{65}\,dt \\
&=\; \sqrt{65}t \quad\longrightarrow\quad t(s) = \frac{s}{\sqrt{65}}
\end{aligned}$$

Therefore, $\mathbf{r}(t(s)) = 8\sin\left(\frac{s}{\sqrt{65}}\right)\mathbf{i} + \left(\frac{s}{\sqrt{65}}\right)\mathbf{j} + \cos\left(\frac{s}{\sqrt{65}}\right)\mathbf{k}$.

(2) [8] Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4 + y^4}$$

Along the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Along the diagonal line $y = x$, $f(x,y) = f(x,x) = 6x^4/3x^4 = 2$ for $x \neq 0$, so along this line $f(x,y) \to 2$. These two limits are different, so the limit asked in the question does not exist.

(3) [10] Find the indicated partial derivatives.

$$f(r,s,t) = r\ln(rs^2t^3); \qquad f_r, \quad f_{ss}, \quad f_{str}$$

$$\begin{aligned}
f_r \;&=\; \ln(rs^2t^3) + r\frac{1}{rs^2t^3}(s^2t^3) \;=\; \ln(rs^2t^3) + 1 \\
f_s \;&=\; \frac{r(2rst^3)}{rs^2t^3} \;=\; \frac{2r}{s} \\
f_{ss} \;&=\; -\frac{2r}{s^2} \\
f_{st} \;&=\; 0 \\
f_{str} \;&=\; 0
\end{aligned}$$

(4) [8] Find the linearization of $z = e^{x^2-y^2}$ at the point $(1,-1,1)$

$f_x(x,y) = 2xe^{x^2-y^2}$, $f_y(x,y) = -2ye^{x^2-y^2}$; so $f_x(1,-1) = 2$, $f_y(1,-1) = 2$. The linearization is the tangent plane; $z - 1 = 2(x-1) + 2(y+1) \quad \rightarrow \quad z = 2x + 2y + 1$.

(5) [8] Use the Chain Rule to find the partial derivatives $\partial w/\partial t$ and $\partial w/\partial s$;

$$w = xe^{y-z^2} \quad \text{where} \quad x = 2st, \quad y = s+t, \quad z = 2t - s$$

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= e^{y-z^2}2s + xe^{y-z^2} - 4xze^{y-z^2} \\
&= (2s + x - 4xz)e^{y-z^2} \;=\; (2s + 2st - 8st(2t-s))e^{s+t-(2t-s)^2} \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= e^{y-z^2}2t + xe^{y-z^2} + 2zxe^{y-z^2} \\
&= (2t + x + 2zx)e^{y-z^2} \;=\; (2t + 2st + 8st(2t-s))e^{s+t-(2t-s)^2}
\end{aligned}$$

(6) [8] Find the directional derivative $\mathbf{D_u}f$ of $f(r,s,\theta) = \dfrac{e^{-r}\sin\theta}{s}$ at the point $(1,1,\pi/3)$ where $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

$$\nabla f = (f_r, f_s, f_\theta) = \left(-\frac{e^{-r}\sin\theta}{s}, -\frac{e^{-r}\sin\theta}{s^2}, \frac{e^{-r}\cos\theta}{s}\right)$$

$$\nabla f(1,1,\pi/3) = \left(\frac{\sqrt{3}}{2e}, \frac{\sqrt{3}}{2e}, \frac{1}{2e}\right)$$

$$\mathbf{D_u}f \;=\; \nabla f(1,1,\pi/3)\cdot\frac{\mathbf{u}}{|\mathbf{u}|} \;=\; \frac{\sqrt{3}}{e\sqrt{14}} - \frac{3\sqrt{3}}{2e\sqrt{14}} + \frac{1}{2e\sqrt{14}}$$

(7) [10] Find the local maximum and minimum values and saddle point(s) of the function

$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$$

in the half-plane $x \geq -2$.

$$\nabla f = (6x^2 + y^2 + 10x, 2xy + 2y) = (0,0) \Longrightarrow (x,y) = (0,0), (-\frac{5}{3}, 0), (-1, \pm 2)$$

$$D(0,0) = 20, f_{xx}(0,0) = 10 \rightarrow f(0,0) \text{ local minimum}$$

$$D(-\frac{5}{3},0) > 0, f_{xx}(-\frac{5}{3},0) < 0 \rightarrow \; f(-\frac{5}{3},0) \text{ local maximim}$$

$$D(-1 \pm 2) < 0 \rightarrow (-1, \pm 2) \text{ saddle points}$$

Along $x = -2$; $f(-2,y) = g(y) = -y^2 + 4 \rightarrow g(y)$ (not $f(x,y)$!) has a local maximum at $y = 0$ (g, and hence f, cannot have a local max or min anywhere else along the line $x = -2$). Now, $\nabla f(-2,0) = (4,0)$ which implies that $\nabla f(-2,0)\cdot\mathbf{u} > 0$ for any unit vector pointing strictly into the half-plane $x \geq -2$ (because if $\mathbf{u} = \langle a,b\rangle$ then $a > 0$). This implies that f is increasing in any direction into $x > -2$ at the point $(-2,0)$. So in fact f does not have a local maximum at $(-2,0)$ in the set $x \geq -2$.