MATH 251 Final examination, (R. Pyke) December, 2006

Time allotted: 180 minutes.
Total Marks: 81. Marks for each question are indicated by [ ]. This exam has 10 questions on 12 pages.

(1) [12] (a) Find the equation of the line of intersection between the two planes

$$z = x + y, \qquad 2x - 5y - z = 0$$

(b) Find the equation of the plane that contains the line $\dfrac{x-3}{2} = y = 8 - z$ and the point $(2, 1, 6)$.

(c) Find the distance from the point $(2, 8, 5)$ and the plane $x - 2y - 2z = 1$

(2) [4] Find the radius of the osculating circle of the helix $\langle \cos t, \sin t, t^2 \rangle$ at $t = \pi/4$.

(3) [4] Find the equation of the tangent plane to the surface

$$\frac{x^2}{4} + y^2 - z^2 = 1$$

above the point $(x, y) = (1, 2)$

(4) [9] (a) Find a <u>formula</u> for the arc length of the graph of $y^2 + 2x^3 = 2$ between the points $(0, \sqrt{2})$ and $(1, 0)$ (do not evaluate the formula).

(b) Find the unit tangent $\mathbf{T}$, normal $\mathbf{N}$, and binormal $\mathbf{B}$ vectors for the curve $\langle 2\sin t,\, 5t,\, 2\cos t\rangle$

(c) Show that $|\mathbf{v} \times \mathbf{N}| = v$ for any smooth curve $\mathbf{r}(t)$ where $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity and $v = |\mathbf{v}(t)|$ is the speed.

(5) [5] You are standing on a mountain whose altitude A in metres at point x km north and y km east of the parking lot is $A(x,y) = 1000+(0.01)e^{-(x^2+4y^2)}[x^2y-255]$. You walk southwest from the parking lot at a horizontal speed of 2m/s. How quickly is your altitude changing 10 minutes later (as you are climbing over the mountain)? (You do not need to simplify your answer.)

(6) [8] Use Lagrange multipliers to find the highest and lowest points on the curve of intersection of the two surfaces

$$4x - 3y + 8z = 5, \qquad z^2 = x^2 + y^2$$

(7) [21] Compute the following integrals.

(a) $\displaystyle\int\int_D e^{-(x^2+y^2)}\,dA$ where D is the cone in the $xy-$plane between the lines $y=x$ and $y=-x$.

(b) $\displaystyle\int\int_D \frac{y}{1+x^2}\,dA$ where D is the (finite) region between the curves $x=y^2$ and $y+x-2=0$.

(c) Find the volume under the surface $z = x^2 y$ and above the triangle with vertices $(1, 0), (2, 1), (4, 0)$.

(d) $\displaystyle\iiint_E z\, dV$ where E is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the quadrant $x \geq 0, y \geq 0, z \geq 0$.

(e) $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dx dy$

(f) $\int\int_R x^2\, dA$ where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$.

(g) Write the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz dy dx$ as an iterated integral $\int\int\int f(x, y, z)\, dx dy dz$ (that is, find the limits of integration in the second integral).

(8) [8] Evaluate the following line integrals.

(a) $\int_C \frac{x}{y}\, dy \;+\; z^2\, dx \;+\; \frac{x+y}{z}\, dz$ where C is the curve $\langle t,\, 2t^2,\, -t^3 \rangle,\ 1 \leq t \leq 4$

(b) $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = xy\mathbf{i} + (x^2 - y)\mathbf{j}$ and C is the part of the graph of $y = x^3 - x$ from $(-1, 0)$ to $(1, 0)$.

(9) [6] For each of the following vector fields $\mathbf{F}$, determine whether $\mathbf{F}$ is conservative or not. If it is, find a potential function for $\mathbf{F}$. If $\mathbf{F}$ is conservative, use this to evaluate the line integral of $\mathbf{F}$ along the curve $\mathbf{r}(t) = \langle 2\cos(2t),\ t^2 - 1 \rangle$ from $t = \pi/6$ to $t = \pi$.

(a) $\mathbf{F} = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$

(b) $\mathbf{F} = \dfrac{x}{x^2+y^2}\,\mathbf{i} + \dfrac{-y}{x^2+y^2}\,\mathbf{j}$

(10) [4] Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C xe^{-2x}\,dx \;+\; (x^4 + 2x^2y^2)\,dy,$$

C is the boundary of the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

END