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Simon Fraser University
Department of Mathematics
Midterm Examination 2
MATH 232
14 November 2005 11:30–12:20

- Please ensure that you sign your exam above to certify your identity. Unsigned exams will not be marked.
- The duration of this exam is 50 minutes.
- DO NOT OPEN this test booklet until told to do so.
- Please check that you have all 6 pages of the exam.
- Do ALL your work in this test booklet. You may use the backside of each page for scrap work.
- The value of each question is shown on the left margins.

Question	Score	Maximum
1		8
2		8
3		8
4		4
5		12
Total		40

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 4 & -1 & 3 & 4 & 0 \\ 1 & 3 & 4 & 1 & 0 \\ 2 & 3 & 4 & 4 & 0 \end{bmatrix}$ and suppose a row echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

[4] (a) Determine a basis for Row(A).

The non-zero rows of B give a basis for Row(A).

[4] (b) Determine a basis for Row(A^t) which consists of rows of A^t.

We note that Row$(A^t) \cong$ Col(A). Columns $1, 2, 3$ of A are pivot columns so they form a basis for Col(A). Hence $\{\begin{bmatrix} 1 & 4 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 3 & 4 & 4 \end{bmatrix}\}$ forms a basis for Row(A^t).

2. Let $V = \mathbb{P}_2$. Consider $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$ and $\mathcal{C} = \{1+t+t^2, 1+t, 2\}$ (where the vectors are ordered as given).

[4] (a) Show that $\mathcal{B}$ is a basis for V.

Let's compute $P_{\mathcal{B}}$. It is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The columns of $P_{\mathcal{B}}$ are linearly-independent and span $\mathbb{R}^3$ as there are 3 pivot columns. Hence, $\mathcal{B}$ is a basis for V.

[4] (b) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

The short way is to note that if $\mathbf{x} = \gamma_1\mathbf{c}_1 + \gamma_2\mathbf{c}_2 + \gamma_3\mathbf{c}_3$ then $\mathbf{x} = 2\gamma_3\mathbf{b}_1 + \gamma_2\mathbf{b}_2 + \gamma_1\mathbf{b}_3$. Hence, $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}$ where

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The longer way is to bring $[P_{\mathcal{B}}|P_{\mathcal{C}}]$ to $\left[I|P_{\mathcal{B}}^{-1}P_{\mathcal{C}}\right]$ using row reduction.

3. Let

$$A(\lambda) = \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}.$$

[4] (a) Show that the determinant of $A(\lambda)$ is $\lambda^2(\lambda^2 - 1)$.

$$\begin{aligned} |A(\lambda)| &= \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} \\ &=_{(R_3 \leftrightarrow R_4)} - \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & -\lambda & 1 \end{vmatrix} \\ &=_{R_4 \leftarrow R_4 + \lambda R_3} - \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 - \lambda^2 \end{vmatrix} \\ &= \lambda^2(\lambda^2 - 1) \end{aligned}$$

[4] (b) Determine a basis for Nul$A(-1)$.

$$A(-1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow_{R_4 \leftarrow R_4 - R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow_{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow_{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, Nul$A(-1) = $ Span $\left\{ \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}^t \right\}$.

[2] **4.** (a) Using the answers from Question 3, determine the characteristic polynomial of

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is given by $\det(B - \lambda I) = \det A(\lambda) = \lambda^2(\lambda^2 - 1)$.

[2] (b) Using the answers from Question 3, determine a basis for the eigenspace of B corresponding to the eigenvalue -1.

The eigenspace of eigenvalue -1 of B is given by Nul$(B+I) =$ Nul$A(-1) =$ Span $\left\{ \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}^t \right\}$.

5. Let V be a vector space over $\mathbb{R}$ with basis $\{\mathbf{b}_1, \mathbf{b}_2\}$. Let $T : V \to \mathbb{R}^2$ and $S : V \to \mathbb{R}^2$ be linear transformations.

[4] (a) Show that $R(\mathbf{x}) = S(\mathbf{x}) + 2T(\mathbf{x})$ is a linear transformation.

$R(\mathbf{x}+\mathbf{y}) = S(\mathbf{x}+\mathbf{y}) + 2T(\mathbf{x}+\mathbf{y}) = S(\mathbf{x}) + S(\mathbf{y}) + 2(T(\mathbf{x}) + T(\mathbf{y})) = S(\mathbf{x}) + 2T(\mathbf{x}) + S(\mathbf{y}) + 2T(\mathbf{y}) = R(\mathbf{x}) + R(\mathbf{y})$. Similarly, $R(c\mathbf{x}) = S(c\mathbf{x}) + 2T(c\mathbf{x}) = c(S(\mathbf{x}) + 2T(\mathbf{x})) = cR(\mathbf{x})$

[4] (b) Suppose that $T(\mathbf{b}_1) = \mathbf{e}_1$, $T(\mathbf{b}_2) = \mathbf{e}_2$, $S(\mathbf{b}_1) = \mathbf{e}_2$, $S(\mathbf{b}_2) = \mathbf{e}_1$. Show that if $\alpha, \beta \in \mathbb{R}$ are such that $\alpha T(\mathbf{x}) + \beta S(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$, then $\alpha = \beta = 0$.

Plugging in $\mathbf{x} = \mathbf{b}_1$ yields $\alpha T(\mathbf{x}) + \beta S(\mathbf{x}) = \alpha \mathbf{e}_1 + \beta \mathbf{e}_2 = \mathbf{0}$. By linear independence of $\mathbf{e}_1, \mathbf{e}_2$, we get that $\alpha = \beta = 0$.

[4] (c) For an invertible matrix A, show that $\det(A^{-1}) = 1/\det(A)$.

Since $AA^{-1} = I$, taking determinants yields $\det A \cdot \det A^{-1} = \det I = 1$. Hence, $\det A^{-1} = 1/\det(A)$.