Math 232, Fall 2007
Midterm 1

Nov. 5, 2007

Last Name:	
First Name:	
SFU ID:	

1. DO NOT LIFT UP THE COVER PAGE UNTIL INSTRUCTED.
2. No calculators are allowed.
3. This test is comprised of 7 pages (including cover page)
4. Once the test begins, please check that all pages are intact.
5. Do ALL questions.
6. Clearly explain your answer. No credit will be given for just writing down the answer.
7. If the answer space provided is not sufficient, write your answer on the back of the previous page. Clearly mark the question number.
8. Good luck.

Question	Points	Score
1	7	
2	10	
3	10	
4	11	
5	12	
Total:	50	

1. (7 points) Compute the determinant of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}.$$

Show all work.

SOLUTION: We use cofactor expansion along the first row to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

Using the formula for the determinant of a 2×2, we get an answer of

$$(18 - 12) - (9 - 3) + (4 - 2) \;=\; 2.$$

So the determinant is 2.

2. True or False. Justify your answers.

   (a) (2 points) If a 3×3 matrix has linearly dependent columns then its columns cannot span $\mathbb{R}^3$.

   TRUE If a square matrix has linearly dependent columns then it is not invertible by the Invertible matrix theorem and so its columns cannot span by the invertible matrix theorem.

   (b) (2 points) If A is a 3×3 invertible matrix, then its determinant must be strictly positive.

   FALSE: Consider the matrix
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (c) (2 points) If the dimension of the Column space of an 8×9 matrix A is 4, then the dimension of its Null space is 5.

   TRUE: The dimension of the column space plus the dimension of the null space is the number of columns of the matrix, in this case 9.

   (d) (2 points) If an invertible matrix A can be row reduced to the identity, then A^{-1} can be row reduced to the identity.

   TRUE: An invertible matrix is always row equivalent to the identity matrix, by the invertible matrix theorem, so this applies to both A and A^{-1}.

   (e) (2 points) If a subset W of $\mathbb{R}^2$ has the property that the sum of any two vectors in W is again in W, then W is a subspace.

   FALSE: Consider the set of vectors in $\mathbb{R}^2$ whose first coordinate is ≥ 0. This is not a subspace because it is not closed under multiplication by scalars. Nevertheless it is closed under addition.

3. (10 points) An economy is divided into two sectors with consumption matrix

$$C = \begin{pmatrix} .4 & .2 \\ .2 & .6 \end{pmatrix}.$$

Find the production level $\vec{x}$ needed to satisfy a final demand

$$\vec{d} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

Show all work.

SOLUTION: We must solve the equation

$$\vec{x} = C\vec{x} + \vec{d}.$$

Equivalently, we need to solve

$$(I - C)\vec{x} = \vec{d}.$$

This solution is given by

$$\vec{x} = (I - C)^{-1}\vec{d}.$$

Note that

$$I - C = \begin{pmatrix} .6 & -.2 \\ -.2 & .4 \end{pmatrix},$$

and so by the formula for an inverse of a 2×2 matrix, we have

$$(I - C)^{-1} = \frac{1}{.2}\begin{pmatrix} .4 & .2 \\ .2 & .6 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

So

$$(I - C)^{-1}\vec{d} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}.$$

4. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 0 & 2 \end{pmatrix}$$

(a) (5 points) Find a basis for $Col(A)$. Show all work.

(b) (6 points) Find a basis for $Nul(A)$.

Solution: To do both parts, we must row reduce A. We first subtract row 1 from row 2 and then subtract twice row 1 from row 3.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 3 & 3 & 0 & 0 \end{pmatrix}.$$

Next we subtract 3 times row 2 from row 3:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & -6 & -6 \end{pmatrix}$$

Notice that columns 1, 2, and 4 are pivot columns. Hence, taking these columns in A, we see

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is a basis for $Col(A)$.

Notice that if we wish to find a basis for $Nul(A)$, we must row reduce the augmented matrix. This is not much different, however, and we obtain:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & -6 & -6 & 0 \end{pmatrix}.$$

If we interpret this as the augmented matrix of a system of equations in unknowns x_1, x_2, x_3, x_4, and x_5, then we see x_3 and x_5 are free. Also, $-6x_4 - 6x_5 = 0$, and so $x_4 = -x_5$; $x_2 + x_3 + 2x_4 + 2x_5 = 0$, and so $x_2 = -x_3$; $x_1 - x_2 + x_5 = 0$, and so $x_1 = -x_3 - x_5$. Hence

$$Nul(A) = \left\{ \begin{bmatrix} -x_3 - x_5 \\ -x_3 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} : x_3, x_5 \in \mathbb{R} \right\}.$$

In parametric form

$$Nul(A) = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Thus

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $Nul(A)$.

5. (12 points) Let W be the subspace of $\mathbb{R}^3$ with basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Find the coordinates vector of

$$\begin{bmatrix} 5h-15 \\ 3h+12 \\ -15 \end{bmatrix}$$

with respect to the basis $\mathcal{B}$. Show all work.

Solution: We form the augmented matrix and row reduce:

$$\begin{pmatrix} 1 & 3 & 5h-15 \\ 2 & -1 & 3h+12 \\ -1 & 2 & -15 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5h-15 \\ 0 & -7 & -7h+42 \\ 0 & 5 & 5h-30 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5h-15 \\ 0 & 1 & h-6 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we analyze. Let x_1, x_2 denote the coordinates. Then the second row tells us that $x_2 = h-6$. Since $x_1 + 3x_2 = 5h-15$, we see $x_1 = 2h+3$. Thus the coordinate vector is

$$\begin{bmatrix} 2h+3 \\ h-6 \end{bmatrix}.$$