Simon Fraser University

Math 232 Midterm 2 Solutions

1. F T F F F T

2.

a. $C_{12} = (-1)^{1+2} \det A_{12} = (-1)^{1+2} \det \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} = 4$

$C_{23} = (-1)^{2+3} \det A_{23} = (-1)^{2+3} \det \begin{pmatrix} 3 & 6 \\ 2 & 3 \end{pmatrix} = 3$

$C_{31} = (-1)^{3+1} \det A_{31} = (-1)^{3+1} \det \begin{pmatrix} 6 & 7 \\ 2 & 2 \end{pmatrix} = -2$

b. $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \dfrac{1}{2} \begin{pmatrix} 2 & -3 & -2 \\ 4 & -2 & -6 \\ -4 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-3}{2} & -1 \\ 2 & -1 & -3 \\ -2 & \frac{3}{2} & 3 \end{pmatrix}$

3.

a. $\det A =_{r_3 \leftarrow r_3 - 2r_1} \det \begin{pmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ -3 & -7 & -5 & 2 \end{pmatrix} =_{r_3 \leftarrow r_3 + r_2} \det \begin{pmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ -3 & -7 & -5 & 2 \end{pmatrix} = 0$

b. $\det A = 0$, so A is not invertible.

4.

a. Because $\mathbb{P}_1$ is isomorphic to $\mathbb{R}^2$ through the standard co-ordinate map (that is, the map that sends the standard basis vectors of $\mathbb{P}_1$ to the standard basis vectors of $\mathbb{R}^2$),

$P_{\mathcal{C} \leftarrow \mathcal{B}} = P_{\mathcal{C}' \leftarrow \mathcal{B}'}$ where $\mathcal{B}' = \left\{ [1+t]_{\mathcal{E}}, [1-t]_{\mathcal{E}} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and

$\mathcal{C}' = \left\{ [2+t]_{\mathcal{E}}, [1+2t]_{\mathcal{E}} \right\} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$, and where $\mathcal{E} = \{1, t\}$ is the standard basis for $\mathbb{P}_1$.

To solve this equivalent problem:

$$(C|B) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 0 & -3 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{1}{3} & -1 \end{array}\right) = \left(I \middle| P_{\mathcal{C}' \leftarrow \mathcal{B}'} \right)$$

And so $P_{\mathcal{C} \leftarrow \mathcal{B}} = P_{\mathcal{C}' \leftarrow \mathcal{B}'} = \begin{pmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & -1 \end{pmatrix}$.

b. $[2]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[2]_{\mathcal{B}} = \begin{pmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{-2}{3} \end{pmatrix}$

5.

a. A set $\mathcal{B}$ to be a **basis** for a vector space *V* if (i) $\mathcal{B}$ is a linearly independent set and (ii) $\mathcal{B}$ is a spanning set for *V*.

b. $\begin{pmatrix} 1 & 1 & -3 & 7 \\ 1 & 2 & -4 & 10 \\ 1 & -1 & -1 & 1 \end{pmatrix} \sim_{\substack{r_2 \leftarrow r_2 - r_1 \\ r_3 \leftarrow r_3 - r_1}} \begin{pmatrix} 1 & 1 & -3 & 7 \\ 0 & 1 & -1 & 3 \\ 0 & -2 & 2 & -6 \end{pmatrix} \sim_{r_3 \leftarrow r_3 + 2r_2} \begin{pmatrix} 1 & 1 & -3 & 7 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim_{r_1 \leftarrow r_1 - r_2} \begin{pmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

From the echelon form we see that the first and second columns of *A* are linearly independent, thus $\mathcal{B}_C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is a basis for Col*A*. A basis for Row*A* consists of the non-zero rows of any echelon form of *A*: $\mathcal{B}_{1,R} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -3 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right\}$ or $\mathcal{B}_{2,R} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right\}$ are two possible choices for a basis for Row*A*. From the reduced echelon form of *A*, the null space may be described as vectors of the form $s\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ -3 \\ 0 \\ 1 \end{pmatrix}$ for any *s* and *t*. Thus

$\mathcal{B}_N = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for Nul*A*.

6.

a. A set *H* to be a **subspace** of *V* if (i) $H \subseteq V$, (ii) $H \neq \varnothing$, (iii) *H* is closed under vector addition, and (iv) *H* is closed under scalar multiplication.

b. *H* is not a subspace of $\mathbb{R}^2$ because, for instance, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in H$ and 2 is a scalar such that $2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \notin H$. If $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ could be written in the form $\begin{pmatrix} a \\ a^2 \end{pmatrix}$, then, equating on the first entry we must have $a = 2$, but then $a^2 = 4$ and not 2 as required.

7.

a. The **kernel** of $T : V \to W$ is the set $\ker T = \left\{ \vec{x} \in V \,\middle|\, T(\vec{x}) = \vec{0} \right\}$.

b. We must verify that ker*T* satisfies the four properties from (6.a) above:

(i) ker*T* is certainly a subset of *V* as the elements of ker*T* are the pre-images of $\vec{0}$ under the transformation *T*.

(ii) Since *T* is a linear transformation, we know that $T\left(\vec{0}\right)=\vec{0}$ and so $\vec{0}\in \ker T$, which is, therefore, non-empty.

(iii) If we choose $\vec{x},\vec{y}\in \ker T$, then $T\left(\vec{x}+\vec{y}\right)=T\left(\vec{x}\right)+T\left(\vec{y}\right)=\vec{0}+\vec{0}=\vec{0}$ and so $\vec{x}+\vec{y}\in \ker T$ (that is, ker*T* is closed under vector addition).

(iv) If $\vec{x}\in \ker T$ and *c* is any scalar, then $T\left(c\vec{x}\right)=cT\left(\vec{x}\right)=c\vec{0}=\vec{0}$ and so $c\vec{x}\in \ker T$ (that is, ker*T* is closed under scalar multiplication).

Taken together, (i) – (iv) mean that ker*T* is a subspace of *V*.