Math 232 Second Midterm Solution March 5, 2007

1. (3 points) Compute the determinant of the following matrix. Show your work.

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$$

Answer

We compute the determinant by expansion with respect to the first row:

$$\det\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix} = 2\det\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 2(4+1) + (1-6) = 5$$

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2. (a) (2 points) Let c be a real number. Compute the determinant of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & c & 0 \\ 1 & 1 & c \end{pmatrix}$$

Answer

Again we expand by first row to get

$$\det(A) = \det \begin{pmatrix} 1 & c \\ 1 & 1 \end{pmatrix} = 1 - c$$

(b) (3 points) For which values of c is A invertible? Give a formula for A^{-1} that is valid for those values of c.

Answer

A square matrix is invertible if and only if its determinant is non-zero, so for A this is the case whenever $c \neq 1$. In those cases, we can use the adjugate formula to compute the inverse of A:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}$$

where $c_{ij} = (-1)^{i+j} \det A_{ij}$ and A_{ij} is A with the i-th row and the j-th column removed. We thus get

$$\begin{array}{lll} c_{11} = \det \begin{pmatrix} c & 0 \\ 1 & c \end{pmatrix} = c^2 & c_{12} = -\det \begin{pmatrix} 1 & 0 \\ 1 & c \end{pmatrix} = -c & c_{13} = \det \begin{pmatrix} 1 & c \\ 1 & 1 \end{pmatrix} = 1 - c \\ c_{21} = -\det \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix} = 1 & c_{22} = \det \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix} = -1 & c_{23} = -\det \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0 \\ c_{31} = \det \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} = -c & c_{32} = -\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1 & c_{33} = \det \begin{pmatrix} 0 & 0 \\ 1 & c \end{pmatrix} = 0 \end{array}$$

and hence

$$A^{-1} = \frac{1}{1-c} \begin{pmatrix} c^2 & 1 & -c \\ -c & -1 & 1 \\ 1-c & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{c^2}{1-c} & \frac{1}{1-c} & \frac{-c}{1-c} \\ \frac{-c}{1-c} & \frac{-1}{1-c} & \frac{1}{1-c} \\ 1 & 0 & 0 \end{pmatrix}$$

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3. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \end{pmatrix}$$

(a) (3 points) Find a basis for Nul(A)

Answer

We row reduce

$$A \sim \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & 2 & -1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \in \mathrm{Nul}(A)$. Furthermore, we see that A has 2 pivots and therefore $\mathrm{rank}(A) = 2$. Since the given 2 vectors are linearly independent, they form a basis.

(b) (3 points) Find a basis for Col(A)

Answer

Since $\mathrm{rank}(A) = 2$, we know that Col(A) is 2-dimensional. Furthermore, the first two columns, $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ lie in the column space and are linearly independent because they correspond to pivot columns. Therefore, these columns form a basis for Col(A).

(c) (1 point) What is the rank of A?

Answer

As remarked, A has 2 pivots in its row echelon form and therefore has rank 2.

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4. We consider the subset

$$H = \{f(t) \in \mathbb{P}_3 : f(-t) = f(t)\}$$

(a) (1 point) What properties should H satisfy to be a subspace of $\mathbb{P}_3$?

Answer

(i) The zero vector (i.e., zero polynomial) should be in H

(ii) If $f_1(t), f_2(t) \in H$ then $f_1(t) + f_2(t) \in H$.

(iii) If $f_1(t) \in H$ and $c \in \mathbb{R}$ then $cf_1(t) \in H$.

(b) (3 points) Prove that H is a subspace of $\mathbb{P}_3$.

Answer

(i) If $f(t) = 0$, the zero polynomial, then $f(t) = 0 = f(-t)$, so the zero polynomial is in H.

(ii) Suppose $f_1(t), f_2(t) \in H$ and let $h(t) = f_1(t) + f_2(t)$. Then

$$h(-t) = f_1(-t) + f_2(-t) = f_1(t) + f_2(t) = h(t)$$

so H is closed under vector addition.

(iii) If $f_1(t) \in H$ and $c \in \mathbb{R}$ then $cf_1(t) \in H$. Suppose $f_1(t) \in H$, $c \in \mathbb{R}$ and let $h(t) = cf(t)$. Then

$$h(-t) = cf(-t) = cf(t) = h(t),$$

so H is closed under scalar multiplication as well.

(c) (3 points) Give a basis of H. Explain why your answer is correct.

Answer

In general, an element of $\mathbb{P}_3$ can be written as $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The $f(t) = f(-t)$ implies that

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3 = a_0 - a_1 t + a_2 t^2 - a_3 t^3,$$

so $a_0 = a_0$, $a_1 = -a_1$, $a_2 = a_2$, $a_3 = -a_3$. In other words, $a_1 = a_3 = 0$. That means that any element of H can be written as a linear combination of 1 and t^2. Since these are linearly independent, we see that $\{1, t^2\}$ is a basis for H.

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5. Consider the subspace

$$H = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \mathbb{R}^3 : x_1 + x_2 = x_3 \right\}$$

(a) (3 points) Show that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for H.

Answer

First we show that $\mathcal{B}$ is linearly independent: If $x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0}$ then $x_1 = 0$ from the first coordinate and $x_2 = 0$ from the second coordinate.
Second, we check that $\mathcal{B} \subset H$: Clearly, the vectors in $\mathcal{B}$ satisfy the equation $x_1 + x_2 = x_3$. Together with the first bit, this establishes that $\dim H \geq 2$ and that if $\dim H = 2$ then $\mathcal{B}$ must span H and hence is a basis.

Third, note that $H \neq \mathbb{R}^3$, because the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin H$. Hence, $\dim H < \dim \mathbb{R}^3 = 3$, so indeed $\dim H = 2$.

(b) (2 points) Describe in words what the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of a vector $\mathbf{v} \in H$ is.

Answer

The coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ is the unique vector $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ such that $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$, where $\mathbf{b}_1, \mathbf{b}_2$ are the elements of $\mathcal{B}$.

(c) (3 points) Let $\mathbf{c}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Compute $[\mathbf{c}_1]_{\mathcal{B}}$ and $[\mathbf{c}_2]_{\mathcal{B}}$.

Answer

Since $\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, we have that $[\mathbf{c}_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Since $\mathbf{c}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, we have that $[\mathbf{c}_2]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(d) (2 points) Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Compute $P_{\mathcal{B} \leftarrow \mathcal{C}}$.

Answer

We have

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} | & | \\ [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
