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Simon Fraser University
Department of Mathematics
Midterm Examination 1
MATH 232
3 October 2005 11:30–12:20

- Please ensure that you sign your exam above to certify your identity. Unsigned exams will not be marked.
- The duration of this exam is 50 minutes.
- DO NOT OPEN this test booklet until told to do so.
- Please check that you have all 6 pages of the exam.
- Do ALL your work in this test booklet. You may use the backside of each page for scrap work.
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Question	Score	Maximum
1		8
2		8
3		6
4		8
5		8
Total		38

- [4] 1. (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}$. Compute the products AB , BA , and write down the augmented matrix of the linear system which results from the matrix equation $AB = BA$.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -a + 2b & -4a + 3b \\ -c + 2d & -4c + 3d \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a - 4c & -b - 4d \\ 2a + 3c & 2b + 3d \end{bmatrix}$$

Equating the entries of AB and BA , we get the linear system

$$\begin{cases} 2b + 4c = 0 \\ -2a - 4c + 2d = 0 \\ -4a + 4b + 4d = 0 \\ -2b - 4c = 0 \end{cases}$$

which has augmented matrix

$$\begin{bmatrix} 0 & 2 & 4 & 0 & 0 \\ -2 & 0 & -4 & 2 & 0 \\ -4 & 4 & 0 & 4 & 0 \\ 0 & -2 & -4 & 0 & 0 \end{bmatrix}.$$

- [4] (b) Bring the following matrix to reduced row echelon form using elementary row operations. Indicate the operations you are performing at each step.

$$\begin{bmatrix} 0 & 2 & 4 & 0 \\ -2 & 0 & -4 & 2 \\ -4 & 4 & 0 & 4 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 0 & 2 & 4 & 0 \\ -2 & 0 & -4 & 2 \\ -4 & 4 & 0 & 4 \\ 0 & -2 & -4 & 0 \end{bmatrix} &\rightarrow \begin{matrix} (R_4 \leftarrow R_4 + R_1) \\ (R_3 \leftarrow R_3 - 2R_2) \end{matrix} \begin{bmatrix} 0 & 2 & 4 & 0 \\ -2 & 0 & -4 & 2 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\rightarrow \begin{matrix} (R_3 \leftarrow R_3 - 2R_1) \\ (R_2 \leftarrow -R_2/2) \end{matrix} \begin{bmatrix} 0 & 2 & 4 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\rightarrow (R_1 \leftrightarrow R_2) \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\rightarrow (R_2 \leftarrow R_2/2) \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

- [4] 2. (a) Write down the solution of the linear system given by the following augmented matrix in vector parametric form.

$$\left[\begin{array}{ccccc} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Suppose the variables are a, b, c, d , respectively. Then $c = s, d = t$ are free. Hence, using the two equations we get that

$$a = 3 - 2s + t$$

$$b = -1 - 2s$$

$$c = s$$

$$d = t.$$

In parametric vector form, this becomes

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- [4] (b) Find a linear system whose solution set is

$$\text{Span} \{ [1, 2, 3, 4]^t, [-1, 2, -3, 4]^t \}.$$

We wish to determine linear conditions on x, y, z, w such that the linear system given by

$$\begin{bmatrix} 1 & -1 & x \\ 2 & 2 & y \\ 3 & -3 & z \\ 4 & 4 & w \end{bmatrix}$$

is consistent. We bring this matrix to RE form to determine consistency conditions.

$$\begin{bmatrix} 1 & -1 & x \\ 2 & 2 & y \\ 3 & -3 & z \\ 4 & 4 & w \end{bmatrix} \xrightarrow{\substack{(R_3 \leftarrow R_3 - 3R_1) \\ (R_4 \leftarrow R_2 - 2R_1)}} \begin{bmatrix} 1 & -1 & x \\ 2 & 2 & y \\ 0 & 0 & z - 3x \\ 0 & 0 & w - 2y \end{bmatrix}$$

$$\xrightarrow{(R_2 \leftarrow R_2 - 2R_1)} \begin{bmatrix} 1 & -1 & x \\ 0 & 4 & y - 2x \\ 0 & 0 & z - 3x \\ 0 & 0 & w - 2y \end{bmatrix}.$$

The linear system is consistent if and only if $z - 3x = 0$ and $w - 2y = 0$. Hence, this linear system has the given span as its solution set.

- [4] 3. (a) Write down the linear transformation $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which rotates points about the y -axis by an angle θ as measured positively from the positive x -axis to the positive z -axis. Show your work by first describing the images of the standard vectors e_1, e_2, e_3 .

The vector e_1 is sent to $[\cos \theta, 0, \sin \theta]^t$. The vector e_2 is sent to $[0, 1, 0]^t$. The vector e_3 is sent to $[-\sin \theta, 0, \cos \theta]^t$. Thus, the matrix is given by

$$\begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

- [2] (b) Write down the linear transformation $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ which sends the vectors $[-1, 0, 0]^t, [0, 1, 0]^t, [0, 0, 2]^t$ to $[-\sqrt{3}/2, -1/2]^t, [\sqrt{3}/2, -1/2]^t, [0, 1]^t$, respectively.

We have that $S(-e_1) = [-\sqrt{3}/2, -1/2]^t$, and hence, $S(e_1) = [\sqrt{3}/2, 1/2]^t$. Also, $S(e_2) = [\sqrt{3}/2, -1/2]^t$. Finally, $S(2e_3) = [0, 1]^t$, so $S(e_3) = [0, 1/2]^t$. Thus, the matrix is given by

$$\begin{bmatrix} \sqrt{3}/2 & \sqrt{3}/2 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

- [4] 4. (a) Let P be the linear transformation with standard matrix

$$A = \begin{bmatrix} -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

Find all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $P(\mathbf{x}) = 0$. Is P onto?

We bring the matrix of P to RE form.

$$\begin{aligned} \begin{bmatrix} -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{bmatrix} &\rightarrow (R_1 \leftarrow -R_1/\sqrt{3}) \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \\ &\rightarrow (R_2 \leftarrow R_2 + R_1) \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix} \end{aligned}$$

Since every row has a leading entry, we see that P is onto.

Let $\mathbf{x} = [x, y, z]^t$. Then the vectors \mathbf{x} such that $P(\mathbf{x}) = 0$ are given by $\text{Span}([1, 1, 1]^t)$

- [4] (b) Let $W = S \circ R$ where $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation and $R(\mathbf{x}) = B\mathbf{x}$ for an invertible 3×3 matrix B . Suppose that $\text{Ker}(S) = \text{Span}\{[1, 1, 1]^t\}$. Find all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $W(\mathbf{x}) = 0$. You need not find the vectors explicitly. It suffices to express the answer in terms of B^{-1} and $\text{Ker}(S)$.

We note that $S(R(\mathbf{x})) = 0$ if and only if $R(\mathbf{x}) \in \text{Span}\{[1, 1, 1]^t\}$ if and only if $\mathbf{x} \in B^{-1}\text{Span}\{[1, 1, 1]^t\} = \text{Span}\{B^{-1}[1, 1, 1]^t\}$.

5. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^m$.

[4] (a) Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2\}$.

We note that $c_1v_1 + c_2v_2 = c_1(v_1 + v_2) + (c_2 - c_1)v_2$. Hence, $\text{Span}\{v_1, v_2\} = \{c_1v_1 + c_2v_2 : c_i \in \mathbb{R}\} \subseteq \text{Span}\{v_1 + v_2, v_2\}$. On the other hand, $c_1(v_1 + v_2) + c_2v_2 = c_1v_1 + (c_1 + c_2)v_2$ so $\text{Span}\{v_1 + v_2, v_2\} = \{c_1(v_1 + v_2) + c_2v_2 : c_i \in \mathbb{R}\} \subseteq \text{Span}\{v_1, v_2\}$

[4] (b) Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly-independent and \mathbf{v}_4 is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly-independent.

Suppose that $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. We cannot have $c_4 \neq 0$ or else v_4 would be a linear combination of v_1, v_2, v_3 . Hence, we have $c_1v_1 + c_2v_2 + c_3v_3 = 0$ and by linear independence of $\{v_1, v_2, v_3\}$ we have that $c_1 = c_2 = c_3 = 0$ as well.