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Simon Fraser University
Department of Mathematics
Midterm Examination 1
MATH 232
3 October 2005 11:30–12:20

Question	Score	Maximum
1		8
2		8
3		6
4		8
5		8
Total		38

- Please ensure that you sign your exam above to certify your identity. Unsigned exams will not be marked.
- The duration of this exam is 50 minutes.
- DO NOT OPEN this test booklet until told to do so.
- Please check that you have all 6 pages of the exam.
- Do ALL your work in this test booklet. You may use the backside of each page for scrap work.
- The value of each question is shown on the left margins.

[4] **1.** (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}$. Compute the products AB, BA, and write down the augmented matrix of the linear system which results from the matrix equation $AB = BA$.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -a+2b & -4a+3b \\ -c+2d & -4c+3d \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a-4c & -b-4d \\ 2a+3c & 2b+3d \end{bmatrix}$$

Equating the entries of AB and BA, we get the linear system

$$\left\{ \begin{array}{rl} 2b+4c & = 0 \\ -2a-4c+2d & = 0 \\ -4a+4b+4d & = 0 \\ -2b-4c & = 0 \end{array} \right\}$$

which has augmented matrix

$$\begin{bmatrix} 0 & 2 & 4 & 0 & 0 \\ -2 & 0 & -4 & 2 & 0 \\ -4 & 4 & 0 & 4 & 0 \\ 0 & -2 & -4 & 0 & 0 \end{bmatrix}.$$

[4] (b) Bring the following matrix to reduced row echelon form using elementary row operations. Indicate the operations you are performing at each step.

$$\begin{bmatrix} 0 & 2 & 4 & 0 \\ -2 & 0 & -4 & 2 \\ -4 & 4 & 0 & 4 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 4 & 0 \\ -2 & 0 & -4 & 2 \\ -4 & 4 & 0 & 4 \\ 0 & -2 & -4 & 0 \end{bmatrix} \longrightarrow \begin{matrix} (R_4 \leftarrow R_4 + R_1) \\ (R_3 \leftarrow R_3 - 2R_2) \end{matrix} \begin{bmatrix} 0 & 2 & 4 & 0 \\ -2 & 0 & -4 & 2 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{matrix} (R_3 \leftarrow R_3 - 2R_1) \\ (R_2 \leftarrow -R_2/2) \end{matrix} \begin{bmatrix} 0 & 2 & 4 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow (R_1 \leftrightarrow R_2) \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow (R_2 \leftarrow R_2/2) \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[4] **2.** (a) Write down the solution of the linear system given by the following augmented matrix in vector parametric form.

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose the variables are a, b, c, d, respectively. Then $c = s, d = t$ are free. Hence, using the two equations we get that

$$\begin{aligned} a &= 3 - 2s + t \\ b &= -1 - 2s \\ c &= s \\ d &= t. \end{aligned}$$

In parametric vector form, this becomes

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

[4] (b) Find a linear system whose solution set is

$$\text{Span}\left\{[1, 2, 3, 4]^t, [-1, 2, -3, 4]^t\right\}.$$

We wish to determine linear conditions on x, y, z, w such that the linear system given by

$$\begin{bmatrix} 1 & -1 & x \\ 2 & 2 & y \\ 3 & -3 & z \\ 4 & 4 & w \end{bmatrix}$$

is consistent. We bring this matrix to RE form to determine consistency conditions.

$$\begin{bmatrix} 1 & -1 & x \\ 2 & 2 & y \\ 3 & -3 & z \\ 4 & 4 & w \end{bmatrix} \longrightarrow \begin{matrix} (R_3 \leftarrow R_3 - 3R_1) \\ (R_4 \leftarrow R_2 - 2R_2) \end{matrix} \begin{bmatrix} 1 & -1 & x \\ 2 & 2 & y \\ 0 & 0 & z - 3x \\ 0 & 0 & w - 2y \end{bmatrix}$$

$$\longrightarrow (R_2 \leftarrow R_2 - 2R_1) \begin{bmatrix} 1 & -1 & x \\ 0 & 4 & y - 2x \\ 0 & 0 & z - 3x \\ 0 & 0 & w - 2y \end{bmatrix}.$$

The linear system is consistent if and only if $z - 3x = 0$ and $w - 2y = 0$. Hence, this linear system has the given span as its solution set.

[4] **3.** (a) Write down the linear transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates points about the y-axis by an angle θ as measured positively from the positive x-axis to the positive z-axis. Show your work by first describing the images of the standard vectors e_1, e_2, e_3.

The vector e_1 is sent to $[\cos\theta, 0, \sin\theta]^t$. The vector e_2 is sent to $[0, 1, 0]^t$. The vector e_3 is sent to $[-\sin\theta, 0, \cos\theta]^t$. Thus, the matrix is given by

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

[2] (b) Write down the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$ which sends the vectors $[-1, 0, 0]^t, [0, 1, 0]^t, [0, 0, 2]^t$ to $[-\sqrt{3}/2, -1/2]^t, [\sqrt{3}/2, -1/2]^t, [0, 1]^t$, respectively.

We have that $S(-e_1) = [-\sqrt{3}/2, -1/2]^t$, and hence, $S(e_1) = [\sqrt{3}/2, 1/2]^t$. Also, $S(e_2) = [\sqrt{3}/2, -1/2]^t$. Finally, $S(2e_3) = [0, 1]^t$, so $S(e_3) = [0, 1/2]^t$. Thus, the matrix is given by

$$\begin{bmatrix} \sqrt{3}/2 & \sqrt{3}/2 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

[4] **4.** (a) Let P be the linear transformation with standard matrix

$$A = \begin{bmatrix} -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

Find all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $P(\mathbf{x}) = 0$. Is P onto?

We bring the matrix of P to RE form.

$$\begin{bmatrix} -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{bmatrix} \longrightarrow \ (R_1 \leftarrow -R_1/\sqrt{3}) \ \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\longrightarrow \ (R_2 \leftarrow R_2 + R_1) \ \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

Since every row has a leading entry, we see that P is onto. Let $\mathbf{x} = [x, y, z]^t$. Then the vectors $\mathbf{x}$ such that $P(\mathbf{x}) = 0$ are given by Span$([1, 1, 1]^t)$

[4] (b) Let $W = S \circ R$ where $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and $R(\mathbf{x}) = B\mathbf{x}$ for an invertible 3×3 matrix B. Suppose that Ker$(S) =$ Span$\{[1, 1, 1]^t\}$. Find all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $W(\mathbf{x}) = 0$. You need not find the vectors explicitly. It suffices to express the answer in terms of B^{-1} and Ker(S).

We note that $S(R(\mathbf{x})) = 0$ if and only if $R(\mathbf{x}) \in$ Span$\{[1, 1, 1]^t\}$ if and only if $\mathbf{x} \in B^{-1}$Span$\{[1, 1, 1]^t\} =$ Span$\left\{B^{-1}[1, 1, 1]^t\right\}$.

5. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^m$.

[4] (a) Show that Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ = Span $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2\}$.

We note that $c_1 v_1 + c_2 v_2 = c_1(v_1 + v_2) + (c_2 - c_1)v_2$. Hence, Span $\{v_1, v_2\} = \{c_1 v_1 + c_2 v_2 : c_i \in \mathbb{R}\} \subseteq$ Span $\{v_1 + v_2, v_2\}$. On the other hand, $c_1(v_1 + v_2) + c_2 v_2 = c_1 v_1 + (c_1 + c_2)v_2$ so Span $\{v_1 + v_2, v_2\} = \{c_1(v_1 + v_2) + c_2 v_2 : c_i \in \mathbb{R}\} \subseteq$ Span $\{v_1, v_2\}$

[4] (b) Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly-independent and $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly-independent.

Suppose that $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 c_4 = 0$. We cannot have $c_4 \neq 0$ or else v_4 would be a linear combination of v_1, v_2, v_3. Hence, we have $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ and by linear indepedence of $\{v_1, v_2, v_3\}$ we have that $c_1 = c_2 = c_3 = 0$ as well.