Math 232, Fall 2007
Midterm 1
Oct. 5, 2007

Last Name:	
First Name:	
SFU ID:	

1. DO NOT LIFT UP THE COVER PAGE UNTIL INSTRUCTED.
2. No calculators are allowed.
3. This test is comprised of 6 pages (including cover page)
4. Once the test begins, please check that all pages are intact.
5. Do ALL questions.
6. Clearly explain your answer. No credit will be given for just writing down the answer.
7. If the answer space provided is not sufficient, write your answer on the back of the previous page. Clearly mark the question number.
8. Good luck.

Question	Points	Score
1	7	
2	10	
3	10	
4	11	
5	12	
Total:	50	

1. (7 points) Show that the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

are linearly independent. Show all work.

SOLUTION: We form the augmented matrix and row reduce.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -2 & 0 \\ 1 & 4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

Since there is a pivot in every column of the coefficient matrix, we see that the vectors are linearly independent.

2. True or False. Justify your answers.

(a) (2 points) If $\vec{v}$ and $\vec{w}$ are two nonzero vectors in $\mathbb{R}^3$ then they are linearly independent.

FALSE If $\vec{v} = \vec{w}$ then they are necessarily linearly dependent.

(b) (2 points) If a transformation T sends $\vec{0}$ to $\vec{0}$ then it is a linear transformation.

FALSE The map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^2$ is not linear.

(c) (2 points) If a row reduced 3×4 matrix has a pivot in every row then its columns span $\mathbb{R}^3$.

TRUE A theorem in class shows that if you have an $m \times n$ matrix with a pivot in every row then the columns span $\mathbb{R}^m$.

(d) (2 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is never onto.

TRUE A linear transformation is onto if and only if its matrix has a pivot in ever row. Since there are 3 rows and only 2 columns, this cannot occur.

(e) (2 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is never one-to-one.

FALSE Consider the map

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

3. (10 points) Balance the chemical equation

$$B_2S_3 + H_2O \longrightarrow H_3BO_3 + H_2S$$

using the vector equation approach. Show all work.

SOLUTION: We create a vector from the components Boron (B), Hydrogen (H), Sulfur (S), and Oxygen (O). We need to find positive integers x_1, x_2, x_3, x_4 that give a solution to the vector equation:

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Taking the vectors on the right hand side over to the left, we get the homogeneous vector equation:

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We now form the augmented matrix and row reduce. We form the augmented matrix and row reduce. Just as in class, we see that the answer is:

$$B_2S_3 + 6H_2O \longrightarrow 2H_3BO_3 + 3H_2S$$

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ 3x_2 + x_1 + 2x_3 \\ x_3 + x_1 \\ x_2 + x_3 \end{bmatrix}$$

(a) (3 points) Find the matrix of T. Show all work.

(b) (4 points) Is T one-to-one? Justify your answer.

(c) (4 points) is T onto? Justify your answer.

Solution: To find the matrix of T, we compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$ and find that the matrix of T is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

To check if T is one-to-one and onto, we row reduce A.

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that this matrix is in echelon form; it has a pivot in every column and therefore T is one-to-one; there is not a pivot in the 4th row, and so T is not onto.

5. Consider the system of linear equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ 2x_1 + 5x_2 + (h-1)x_3 &= -1 \\ 3x_1 + (6-h)x_2 + (-h-4)x_3 &= -4 \end{aligned}$$

For which values of h does the equation have:

(a) (3 points) For which values of h does the system have no solutions?

(b) (3 points) For which values of h does the system have exactly one solution?

(c) (3 points) For which values of h does the equation have exactly two solutions?

(d) (3 points) For which values of h does the equation have infinitely many solutions?

Justify your answers and show all work.

Solution: We form the augmented matrix and row reduce:

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 5 & (h-1) & -1 \\ 3 & 6-h & -h-4 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & h+1 & 1 \\ 0 & -h & -h-1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 & -1 \\ 0 & 1 & h+1 & 1 \\ 0 & 0 & h^2-1 & h-1 \end{pmatrix}$$

Now we analyze.

No solutions: There are no solutions if and only if there is a row of the form $(0\ 0\ 0\ b)$ with $b \neq 0$. The only way this can occur is if $h^2 - 1 = 0$ and $h - 1 \neq 0$. Since $h^2 - 1 = (h-1)(h+1)$ and $h - 1$ must be nonzero, we see that the only way to get no solutions is if $h = -1$.

One solution. To get one solution, we need to have a pivot in every column and no rows of the form $(0\ 0\ 0\ b)$ with $b \neq 0$. This happens if and only if $h^2 - 1 \neq 0$, so $h \neq 1, -1$ gives exactly one solution.

Two solutions: This can never occur. There is either zero, one, or infinitely many solutions.

Infinitely many solutions: This happens if we have a free variable and no rows of the form $(0\ 0\ 0\ b)$ with $b \neq 0$. This happens when $h = 1$.