Simon Fraser University

Math 232 Midterm 1 Solutions

1. F T T T F T

2. $AB = \begin{pmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{pmatrix}$. For BA to be defined, the number of columns of B would have to equal the number of rows of A. This is not the case.

3.

a. $\left[A|b\right] = \left[\begin{array}{ccc|c} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 4 \end{array}\right]$

b. $\left[A|b\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -7 & 13 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

This means that $\begin{cases} x_1 = 13 + 7x_3 \\ x_2 = -5 - 4x_3 \\ x_3 \in \mathbb{R} \end{cases}$, or that $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix}$.

c. $\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = -7\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$, and so the set $\left\{\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}\right\}$ is not linearly independent.

4.

a. There is no pivot in column 3. This means that the matrix is not row-equivalent to the identity matrix, and by the Invertible Matrix Theorem, it is not invertible.

b. $\left[A|I\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -4 & 1 & 3 & 0 & 1 & 0 \\ 3 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 5 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 3 & 5 & 2 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{17}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{3} & \frac{2}{3} & \frac{1}{3} \end{array}\right]$

And so $A^{-1} = \begin{pmatrix} \frac{8}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{17}{3} & \frac{5}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

5.

a. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **linear** if

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$T(c\vec{u}) = cT(\vec{u})$$

for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$.

b. $$T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} = \begin{pmatrix} 3(u_1+v_1)-5(u_2+v_2) \\ (u_1+v_1)+2(u_2+v_2) \end{pmatrix} = \begin{pmatrix} 3u_1-5u_2 \\ u_1+2u_2 \end{pmatrix} + \begin{pmatrix} 3v_1-5v_2 \\ v_1+2v_2 \end{pmatrix}$$
$$= T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$T\left(c\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = T\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} = \begin{pmatrix} 3(cu_1)-5(cu_2) \\ (cu_1)+2(cu_2) \end{pmatrix} = c\begin{pmatrix} 3u_1-5u_2 \\ u_1+2u_2 \end{pmatrix} = cT\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Thus T satisfies the definition in (a), and so T is linear.

c. $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$

6.

a. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if whenever $\vec{u} \neq \vec{v}$, it follows that $T(\vec{u}) \neq T(\vec{v})$.

b. For sufficiency, suppose that $T(\vec{x}) = \vec{0}$ has a non-trivial solution, $\vec{x} = \vec{c}$. As T is linear, $\vec{x} = \vec{0}$ is always a solution to $T(\vec{x}) = \vec{0}$. Since $\vec{c} \neq \vec{0}$ and $T(\vec{c}) = T(\vec{0}) = \vec{0}$, T fails to satisfy the definition of one-to-one.

For necessity, suppose that T is not one-to-one. Then there are vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ where $\vec{a} \neq \vec{b}$, but $T(\vec{a}) = \vec{c} \in \mathbb{R}^m$ and $T(\vec{b}) = \vec{c}$. As $\vec{a} \neq \vec{b}$, we have that $\vec{a} - \vec{b} \neq \vec{0}$. Now

$$T(\vec{a}-\vec{b}) = T(\vec{a}) - T(\vec{b}) = \vec{c} - \vec{c} = \vec{0}$$

and so $T(\vec{x}) = \vec{0}$ has a non-trivial solution.