Math 232 First Midterm Solution February 5, 2007

1. We consider the following system of equations:

$$\begin{cases} x_1 & +x_2 & +2x_3 & -4x_4 & = & 1 \\ x_1 & +2x_2 & +x_3 & +x_4 & = & 2 \\ 2x_1 & +4x_2 & +2x_3 & -x_4 & = & 1 \end{cases}$$

   (a) (1 point) Write down the augmented matrix corresponding to this system.

Answer

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 2 & -1 & 1 \end{array}\right)$$

   (b) (3 points) Determine the reduced row echelon form of this matrix. Show your work. (use the back of the previous page if you need more room)

Answer

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 1 \\ 0 & 1 & -1 & 5 & 1 \\ 0 & 2 & -2 & 7 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 1 \\ 0 & 1 & -1 & 5 & 1 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 1 \\ 0 & 1 & -1 & 5 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & -1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

(c) (2 points) Write down the full solution set to the system in parametric form.

Answer

From part (b) we see that the given system is equivalent to

$$\begin{cases} x_1 &= 9 - 3x_3 \\ x_2 &= -4 + x_3 \\ x_3 & \text{free variable} \\ x_4 &= 1 \end{cases}$$

Hence, in parametric form, the set of solution vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is:

$$\left\{ \begin{pmatrix} 9 \\ -4 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$

(d) (2 points) Let $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$. Can you write $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_4$? Motivate your answer and, if it is "yes", give such a linear combination. [HINT: Relate the given vectors to the system given in (a)]

Answer

In vector notation, the system in this question is

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b}$$

hence, any solution $(x_1, x_2, x_3, x_4$ to this system gives a way of expressing $\mathbf{b}$ as a linear combination of the $\mathbf{a}_i$. We have determined all solutions in (c), so we can for instance write:

$$9\mathbf{a}_1 - 4\mathbf{a}_2 + 0\mathbf{a}_3 + 1\mathbf{a}_4 = \mathbf{b}$$

2. (a) (2 points) Give the definition of *linear independence* for a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of vectors in $\mathbb{R}^n$. Your answer should start with:
"A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is called linearly independent if…"

Answer

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is called linearly independent if the only way to express the zero-vector $\mathbf{0}$ as linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is by taking all weights 0, i.e., the only solution to the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_r\mathbf{v}_r = \mathbf{0}$$

is $x_1 = x_2 = \cdots = x_r = 0$.

(b) (3 points) Can a set of r vectors in $\mathbb{R}^n$ ever be linearly independent if $n < r$? Prove your statement.

Answer

No. As stated above, the vector equation $x_1\mathbf{v}_1 + \cdots + x_r\mathbf{v}_r = \mathbf{0}$ would have to have a unique solution, i.e., *no free variables.*

However, the matrix corresponding to this system would have n rows and r columns. The row echelon form has at most one pivot per row. Since $r > n$, there are more columns than rows, so there is a column without a pivot. This corresponds to a free variable.

(c) (4 points) Determine if the set $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} \right\}$ is linearly independent. Show that your answer is correct.

Answer

We test if the equation

$$x_1 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution:

$$\begin{pmatrix} 1 & 1 & 6 \\ 3 & -1 & 2 \\ 2 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 6 \\ 0 & -4 & -16 \\ 0 & -2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that the corresponding echelon form does not have a pivot in every column, the system does not have a unique solution and therefore the set is not linearly independent.

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3. (a) (4 points) Determine the inverse of the matrix. Show your work.

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Answer

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right)$$

Hence,

$$M^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

(b) (3 points) Prove that, for an invertible $n \times n$ matrix A, the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$, is onto.

Answer

A transformation is called *onto* if for any vector $\mathbf{b} \in \mathbb{R}^n$ (the codomain), we can find a vector $\mathbf{x}$ in the domain such that $T(\mathbf{x}) = \mathbf{b}$. Therefore, in order to show that T is onto, we need to show that for any $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

If A is invertible, we can take $\mathbf{x} = A^{-1}\mathbf{b}$. Then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = AA^{-1}\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$, so indeed T is onto if A is invertible.

4. Mark true or false and give a reason.

   (a) (2 points) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can never be one-to-one if $n > m$.

Answer

True. Let A be the standard matrix of T, so that $T(\mathbf{x}) = A\mathbf{x}$. The matrix A will be $m \times n$. If $n > m$, this means that the row echelon form of A will have columns without pivot. Hence, the system $A\mathbf{x} = \mathbf{0}$ has free variables and therefore a non-trivial solution, say $\mathbf{v} \neq \mathbf{0}$. It follows that $T(\mathbf{v}) = A\mathbf{v} = \mathbf{0}$ and $T(\mathbf{0}) = A\mathbf{0} = \mathbf{0}$. It follows that two distinct vectors have the same image under T and therefore T is not one-to-one.

(b) (2 points) The following map is linear:

$$\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \mapsto & \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} \end{array}$$

Answer

True. We call the transformation T. We need to show that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $c \in \mathbb{R}$ we have $T(\mathbf{v} + bw) = T(\mathbf{v}) + T(\mathbf{w})$ and $T(c\mathbf{v}) = cT(\mathbf{v})$:

$$T(\mathbf{v}+\mathbf{w}) = T\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} = \begin{pmatrix} (v_1 + w_1) + (v_2 + w_2) \\ (v_1 + w_1) - (v_2 + w_2) \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_1 - v_2 \end{pmatrix} + \begin{pmatrix} w_1 + w_2 \\ w_1 - w_2 \end{pmatrix} = T(\mathbf{v}) + T(\mathbf{w})$$

$$T(c\mathbf{v}) = T\begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix} = \begin{pmatrix} cv_1 + cv_2 \\ cv_1 - cv_2 \end{pmatrix} = c\begin{pmatrix} v_1 + v_2 \\ v_1 - v_2 \end{pmatrix} = cT(\mathbf{v})$$

(c) (2 points) A matrix transformation is always an linear transformation.

Answer

True. A matrix transformation is of the form $T(\mathbf{x}) = A\mathbf{x}$. The fact that $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$ and $A(c\mathbf{v}) = cA\mathbf{v}$ are properties of matrix multiplication.
