MATH 232 Final Exam
August 7th, 2001

Time allowed is 3 hours. There are 19 questions, one per page. Attempt all questions. The total number of marks is 150. A one page sheet of handwritten notes (on both sides) is allowed in the exam. Calculators are not permitted. Put your answers in the space provided. If you need more space, use the back of a page. Please fill in your

Full Name: ....

Student ID number: ....

Signature: ....

Question 1 (10 marks)

Answer true (T) if the following statements are *always* true and false (F) otherwise. You will receive one mark for each correct answer, no mark for no answer and lose one mark for each incorrect answer.

In the following $u, v \in \mathbb{R}^n$ and $A, B, C \in \mathbb{R}^{n \times n}$.

(a) If $||u|| = 0$ then $u = \mathbf{0}$.

(b) $||u - v|| \leq ||u|| - ||v||$.

(c) $||u + v|| \leq ||u|| + ||v||$.

(d) $AB = BA$

(e) $A(BC) = (AB)C$

(f) If $AB = AC$ then $B = C$.

(g) If A is singular then the linear system $Ax = u$ has no solution.

(h) If A is invertible then the column vectors of A are linearly independent.

(i) A linear system with the same number of equations as unknowns has a unique solution.

(j) A linear system with more equations than unknowns has no solution.

Question 2 (6 marks)

Let $u = [1, 2]$ be a vector in $\mathbb{R}^2$. Find a non-zero vector v in $\mathbb{R}^2$ which is perpendicular (orthogonal) to u. Sketch u, v and $u+v$ and calculate the area of the square with vertices $[0, 0], u, v$ and $u + v$.

Question 3 (8 marks)

Consider the following linear system in x_1, x_2, x_3, x_4.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1, \\ 3x_1 + 6x_2 - 8x_3 + x_4 &= 2. \end{aligned}$$

Write it in the form $Ax = b$. Solve it by reducing the augmented matrix $[A|b]$ to REDUCED row Echelon form. Express the solutions as a SET of vectors or points in $\mathbb{R}^4$.

Question 4 (6 marks)

Let $u = [1, 0, 1], v = [1, 2, 3]$ and $w = [0, 1, 1]$ be vectors in $\mathbb{R}^3$. Determine whether or not $u \in \mathrm{sp}(v, w)$ by setting up a linear system in the form $Ax = b$ and determining whether it is consistent or not. Show your working.

Question 5 (8 marks)

A square matrix A is said to be nilpotent if $A^r = \mathbf{0}$ for some integer $r > 0$. Show that an invertible matrix cannot be nilpotent.

Question 6 (10 marks)

Answer true (T) if the following statements are *always* true and false (F) otherwise. You will receive one mark for each correct answer, no mark for no answer, and lose a mark for each incorrect answer.

Let V be a (real) Vector space and let u, v and w be distinct vectors in V.

(a) $u + v = v + u$.

(b) Every subspace of V contains the zero vector of V.

(c) If W is a subspace of V and $u, v \in W$ then $u - v \in W$.

(d) $\mathrm{sp}(u, v + w)$ is a subspace of V.

(e) The set union of two subspaces of V is a subspace of V.

(f) If $\{u, v\}$ is a basis for V then $\{u, v, w\}$ cannot be a basis for V.

(g) If $\{u, v, w\}$ is linearly independent then $\dim(V) = 3$.

(h) If $\dim(V) = 2$ then $\{u, v, w\}$ is linearly dependent.

(i) If $\dim(V) = 3$ then $\{u, v, w\}$ is a basis for V.

(j) If $\mathrm{sp}(u, v, w) = V$ then every vector in V can be expressed as a linear combination of u, v, w.

Question 7 (6 marks)

Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose $T([1,1]) = [1,-1]$ and $T([1,0]) = [-1,0]$. Determine the standard matrix representation for T, i.e. find the matrix A such that $T(x) = Ax$. Show your working.

Question 8 (9 marks)

Let $U = \mathrm{sp}([1, -1, 0], [0, -1, 1])$ and $V = \mathrm{sp}([1, 1, 0], [0, 1, 1])$ be subspaces of $\mathbb{R}^3$ and let $W = U \cap V$.
What kind of geometric objects are U, V and W?
Find a basis for W using any method.

Question 9 (6 marks)

Let $B = \{u, v, w\}$ where $u = x^2 - 1, v = x - 1$, and $w = x^2 + x$.
Show that B is a basis for the vector space $\mathbb{R}[x]/x^3 = \mathrm{sp}(1, x, x^2)$.

Question 10 (8 marks)

Let U and V be subspaces of a vector space W.
Prove that $U \cap V$ is a subspace of W.

Question 11 (10 marks)

Answer true (T) if the following statements are *always* true and false (F) otherwise. You will receive one mark for each correct answer, no mark for no answer, and lose a mark for each incorrect answer.

Let A and B be n by n matrices over $\mathbb{R}$.

(a) If A is invertible then $\det(A) = 0$.

(b) If A is row equivalent to B then $\det(A) = \det(B)$.

(c) If A is invertible then $\det(B) = \det(A^{-1}\,B\,A)$.

(d) $\det(I) = 0$.

(e) $\det(A\,B) \neq 0 \implies A$ and B are invertible.

(f) $\det(2\,A) = 2^n \det(A)$.

(g) If $\det(A) = \det(B)$ then $A = B$.

(h) $\det(A\,B) = \det(A)\det(B)$.

(i) If λ is an eigenvalue of A then 2λ is an eigenvalue of A.

(j) If v is an eigenvector of A then $2v$ is an eigenvector of A.

Question 12 (8 marks)

Calculate the determinant of the following matrix, firstly, by expanding along a row or down a column (your choice), AND secondly, by using Gaussian elimination to reduce A to row Echelon form. Show your working.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$$

Question 13 (8 marks)

Define the eigenvalues and eigenvectors of square matrix A.
Give an example of a 2 by 2 matrix A over $\mathbb{R}$ which has complex eigenvalues.
What are its eigenvalues?

Question 14 (6 marks)

Suppose we model a population growth using the Leslie matrix L with two age groups G_1 and G_2. Let F_1 and F_2 be the fertility rates and let P_1 and P_2 be are the survival proportions of the two age groups. Suppose $F_1 = 1/4$, $F_2 = 1/2$, $P_1 = 1/4$, and $P_2 = 1/2$, that is, the Leslie matrix

$$L = \begin{bmatrix} 1/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}.$$

Calculate the eigenvalues of L.
Will the population decline in the long term or grow in the long term?
What will the long term population age distribution vector be?

Question 15 (8 marks)

Let $A \in \mathbb{R}^{n \times n}$. Prove that $\det(A) = 0$ if and only if A has a zero eigenvalue.

Question 16 (9 marks)

Let $W = \mathrm{sp}([1, 1, 1])$. Thus W is a subspace of $\mathbb{R}^3$.
Calculate a basis for $W^\perp$ the orthogonal complement of W.
Find b_W the projection of the vector $b = [3, 1, 2]$ onto W.
Find also $b_{W^\perp}$ the projection of b onto $W^\perp$.

Question 17 (6 marks)

Let $B = \{u, v, w\}$ be a basis for a subspace W of $\mathbb{R}^n$. Suppose u is orthogonal to v. Give a formula for the projection of w onto $\mathrm{sp}(u, v)$. Now give an orthogonal basis for W which contains u and v.

Question 18 (8 marks)

Prove that the projection matrix P is symmetric and idempotent. You may assume $P = A(A^T A)^{-1} A^T$ where A is an n by k matrix whose columns form a basis for a subspace W of $\mathbb{R}^n$.

Question 19 (10 marks)

Answer true (T) if the following statements are true and false (F) otherwise. You will receive one mark for each correct answer, no mark for no answer and lose one mark for each incorrect answer.

In the following $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

(a) A is idempotent.

(b) A is symmetric.

(c) A is nilpotent.

(d) A is invertible.

(e) A is row equivalent to I.

(f) A has nullspace $\{\mathbf{0}\}$.

(g) A has rank 2.

(h) A has a zero eigenvalue.

(i) A is diagonalizable.

(j) A is a projection matrix.

THE END