SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

Final Exam

MATH 232

December 15, 1999, 3:30 – 6:30 p.m.

Name: ______________________________

family name *given name*

Number: ______________________________

INSTRUCTIONS

1. This exam has 14 questions on 16 pages. Please check to make sure your exam is complete.
2. Write your final answer in the answer box.
3. In each question indicate how you obtain your answer. **You may lose points if your work is poorly presented.**
4. If you need more room, use the reverse side of the **previous page** to show your work.
5. No calculators or other computing devices may be used.
6. Please write with a black or blue pen.

Question	Score	Max
1		6
2		6
3		6
4		7
5		5
6		6
7		10
8		8
9		4
10		11
11		9
12		8
13		7
14		7
Total		100

[4] **1.** (a) **Give a precise description of the kinds of row operation which are permitted in bringing a matrix to reduced row-echelon form.**

ANSWER

[2] (b) **Define an elementary matrix.**

ANSWER

ROUGH WORK IF REQUIRED

[6] **2. Find a basis for the set of solutions to the system**

$$\begin{array}{rcrcrcrcl} 2x_1 & - & x_2 & - & 6x_3 & + & 10x_4 & = & 0 \\ -x_1 & + & 3x_2 & + & 8x_3 & - & 15x_4 & = & 0\,. \end{array}$$

ANSWER

SHOW YOUR WORK

3. Let $V = \text{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5)$ denote the subspace of $\mathbb{R}^5$ spanned by

$$\begin{aligned} \boldsymbol{a}_1 &= [2, 2, 1, -1, 0] \\ \boldsymbol{a}_2 &= [-1, 1, 1, 2, 2] \\ \boldsymbol{a}_3 &= [7, 1, -1, -8, -6] \\ \boldsymbol{a}_4 &= [0, 8, 6, 6, 8] \\ \boldsymbol{a}_5 &= [1, 1, 1, -1, -3] \end{aligned}$$

and $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \end{bmatrix}$ be the 5×5 matrix whose columns are $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$, $\boldsymbol{a}_5$.

By elementary row operations A is converted to

$$H = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

[2] (a) **Write down a basis for V.**

ANSWER

[2] (b) **Write down a basis for the row space of A.**

ANSWER

[2] (c) **Determine the rank of A. Give a reason for your answer.**

ANSWER

ROUGH WORK IF REQUIRED

(use the back of page 2)

[7] **4.** On a separate sheet circulated with this exam you find the definition of a *vector space over* $\mathbb{R}$.

Let V be a vector space over $\mathbb{R}$. ¿From the axioms listed in the definition, prove that, for any two vectors v and w in V there exists a unique vector x in V such that $v + x = w$.

ANSWER

ROUGH WORK

[5] **5.** **Let V be a vector space over $\mathbb{R}$. Let W_1 and W_2 be two subspaces of V. Prove that their intersection $W_1 \cap W_2$ is a subspace of V.**

ANSWER

ROUGH WORK

[6] **6.** Let $\mathbb{R}^{2\times 2}$ denote the vector space of all 2×2 real matrices, using as vector addition and scalar multiplication the usual addition of matrices and multiplication of a matrix by a scalar.

Given are four matrices

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}, \quad \boldsymbol{v}_4 = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}.$$

It is given that $\mathcal{B} = (\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4)$ is an ordered basis for $\mathbb{R}^{2\times 2}$.

Let

$$\boldsymbol{v} = \begin{bmatrix} 2 & -1 \\ 6 & 6 \end{bmatrix}.$$

Find the coordinate vector $v_{\mathcal{B}}$ of v relative to $\mathcal{B}$.

ANSWER

SHOW YOUR WORK

7. Let F be the vector space of all functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let W be the subspace of F spanned by the four functions 1, x, e^x and xe^x. It is given that $\mathcal{B} = (1, x, e^x, xe^x)$ is an ordered basis for W.

Given are two linear transformations $T_1 : W \to W$ and $T_2 : W \to W$ defined by

$T_1(f) = f'$ (the derivative of f with respect to x) for all $f \in W$

$T_2(f) = f''$ (the second derivative of f with respect to x) for all $f \in W$.

Let A_1 be the matrix representation of T_1 relative to $\mathcal{B}, \mathcal{B}$ and let A_2 be the matrix representation of T_2 relative to $\mathcal{B}, \mathcal{B}$.

[5] (a) **Find the matrix A_1.**

ANSWER

SHOW YOUR WORK

(Question **7.** continues here.)

[3] (b) **Decide whether the transformation T_1 is invertible. Justify your answer.**

ANSWER

[2] (c) **Use the composition of linear transformations to discover a simple relation between A_2 and A_1. Justify your answer. Do not compute A_2 explicitly, just express it in terms of A_1.**

ANSWER

ROUGH WORK IF REQUIRED

[4] **8.** (a) **Given are three points $P = (3, -1)$, $Q = (2, 2)$ and $R = (-1, 7)$.**
Find the area of the triangle PQR.

ANSWER

[4] (b) **State the row-interchange property for determinants of square matrices.**
Use it to prove:

If two rows of a square matrix A are equal, then $\det(A) = 0$.

ANSWER

SHOW YOUR WORK

[4] **9. Evaluate the determinant**

$$\begin{vmatrix} 0 & 3 & 3 & 5 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 3 & -4 \\ -2 & 0 & 1 & 7 \end{vmatrix}.$$

ANSWER

SHOW YOUR WORK

10. Let

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}.$$

[5] (a) **Find the eigenvalues and corresponding eigenspaces of A.**

ANSWER

SHOW YOUR WORK

(Question **10.** continues here.)

[3] (b) **Use diagonalization to compute A^{2000}. Give your answer in the form of a single 3×3 matrix.**

ANSWER

[3] (c) **Decide whether the matrix**

$$B = \begin{bmatrix} 4 & 3 \\ 0 & 4 \end{bmatrix}$$

is diagonalizable. Justify your answer.

ANSWER

SHOW YOUR WORK

[2] **11.** (a) **Let $\boldsymbol{a} = [2, 1, -1]$ and $\boldsymbol{b} = [-1, 3, 0]$. Find the projection of $\boldsymbol{b}$ on $\mathrm{sp}(\boldsymbol{a})$.**

ANSWER

[3] (b) **Let W be the subspace of $\mathbb{R}^3$ defined by**

$$W = \{[x, y, z] \in \mathbb{R}^3 \mid x + y - z = 0\}.$$

Write down the basis for $W^{\perp}$, the orthogonal complement of W.

ANSWER

[4] (c) **Let $\boldsymbol{c} = [2, 1, 6]$. Find the projection of $\boldsymbol{c}$ on W.**

ANSWER

SHOW YOUR WORK *(use the back of the previous page if necessary)*

12. Let $V = \text{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $\boldsymbol{a}_1 = [1, 0, 0, 1]$, $\boldsymbol{a}_2 = [1, 1, 0, 1]$ and $\boldsymbol{a}_3 = [0, 1, -1, 0]$.

[6] (a) **Find an orthogonal basis for V.**

ANSWER

[2] (b) **Use your answer to part (a) to find an orthonormal basis for V.**

ANSWER

SHOW YOUR WORK *(use the back of the previous page if necessary)*

[7] **13. The following data points are given:**

$$(-2, 0),\ (-1, 1),\ (0, 3),\ (1, 6)\,.$$

Find the least-squares linear fit for these data points.

ANSWER

SHOW YOUR WORK

14. Let $E = (\boldsymbol{e}_1, \boldsymbol{e}_2)$ be the standard ordered basis for $\mathbb{R}^2$. Let $\boldsymbol{b}_1 = [2, 1]$, $\boldsymbol{b}_2 = [-3, -2]$ and let $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ be an ordered basis for $\mathbb{R}^2$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T([x_1, x_2]) = [x_1 + x_2,\ x_1 - x_2]$ for every $[x_1, x_2] \in \mathbb{R}^2$.

[2] (a) **Write down the standard matrix representation of T.**

ANSWER

[2] (b) **Find the change-of-coordinates matrix from E to B.**

ANSWER

[3] (c) **Find the matrix representation of T relative to B.**

ANSWER

SHOW YOUR WORK