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Simon Fraser University
Department of Mathematics
Final Examination
MATH 232
12 December 2005 8:30am – 11:30am

- The duration of this exam is 3 hours.
- DO NOT OPEN this test booklet until told to do so.
- Please check that you have all 10 pages of the exam.
- Do ALL your work in this test booklet.
- The value of each question is shown on the left margins.

| Question | Score | Maximum |
|---|---|---|
| 1 | | 4 |
| 2 | | 8 |
| 3 | | 10 |
| 4 | | 12 |
| 5 | | 10 |
| 6 | | 6 |
| 7 | | 6 |
| 8 | | 10 |
| Total | | 66 |

[4] **1.** Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Give the augmented matrix of the linear system which results from the matrix equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.

We have that

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ A^T\mathbf{b} &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The augmented matrix of the associated linear system is thus

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

[2] **2.** (a) Give an example of a $3 \times 4$ matrix $A_1$ in row echelon form such that $A_1\mathbf{x} = \mathbf{b}_1$ has a solution for every $\mathbf{b}_1 \in \mathbb{R}^3$. Justify your answer.

Let $A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. There is a pivot entry in every row so the linear system $A_1\mathbf{x} = \mathbf{b}_1$ has a solution for every $\mathbf{b}_1 \in \mathbb{R}^3$.

[2] (b) For a given $\mathbf{b}_1 \in \mathbb{R}^3$, must a solution $\mathbf{x}$ to $A_1\mathbf{x} = \mathbf{b}_1$ be unique?

The solutions to $A_1\mathbf{x} = \mathbf{b}_1$ are not unique as the dimension of the null space of $A_1$ is $1$.

[4] (c) Give an example of a $3 \times 4$ matrix $A_2$ in row echelon form and vectors $\mathbf{b}_2, \mathbf{b}_2' \in \mathbb{R}^3$ such that $A_2\mathbf{x} = \mathbf{b}_2$ does not have a solution, whereas $A_2\mathbf{x} = \mathbf{b}_2'$ does have a solution.

Let $A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. For $\mathbf{b}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, the linear system $A_2\mathbf{x} = \mathbf{b}_2$ has a solution, namely $\mathbf{x} = \mathbf{0}$. For $\mathbf{b}_2' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, the linear system $A_2\mathbf{x} = \mathbf{b}_2'$ is inconsistent.

**3.** Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix}$. Suppose that $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ are eigenvectors for $A$.

[2] (a) What are the eigenvalues corresponding to each of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Hence the eigenvalues of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are $1, 2, 2$ respectively.

[4] (b) Is $A$ diagonalizable? If so, give an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

$A$ is diagonalizable as $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$. The matrix $P$ and $D$ are given by

$$P = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$
$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

[4] (c) Let $\lambda_2$ be the eigenvalue corresponding to the eigenvector $\mathbf{b}_2$ of $A$. Find a basis for the eigenspace of $A$ corresponding to the eigenvalue $\lambda_2$. Justify your answer carefully.

Note that $\lambda_2 = 2$ and both $\mathbf{b}_2, \mathbf{b}_3 \in E_A(2)$ and are linearly-independent. We cannot have that $E_A(2) = \mathbb{R}^3$ as there is an eigenvector with eigenvalue $1$, namely $\mathbf{b}_1$. Hence, $\dim E_A(2) = 2$ and $\{\mathbf{b}_1, \mathbf{b}_2\}$ forms a basis for $E_A(2)$.

**4.** Let $V$ be an inner product space. Let $W$ be a subspace of $V$.

[4] (a) Prove that $W^{\perp}$ is a subspace of $V$.

Note that $\mathbf{0} \in W^{\perp}$ as $< \mathbf{0}, \mathbf{w} >= 0$ for all $\mathbf{w} \in W$. Suppose $\mathbf{u}, \mathbf{v} \in W^{\perp}$ and $c \in \mathbb{R}$. Then $< \mathbf{u} + \mathbf{v}), \mathbf{w} >=< \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} >= 0$ and $< c\mathbf{u}), \mathbf{w} >= c < \mathbf{u}, \mathbf{w} >= 0$ for all $\mathbf{w} \in W$ as required.

[4] (b) Suppose $\mathbf{w} \in W, \mathbf{w}^{\perp} \in W^{\perp}$, $\mathbf{w} \neq \mathbf{0}$, $\mathbf{w}^{\perp} \neq \mathbf{0}$, and $\alpha\mathbf{w} + \beta\mathbf{w}^{\perp} = \mathbf{0}$. Prove that $\alpha = \beta = 0$.

Suppose $\alpha\mathbf{w} + \beta\mathbf{w}^{\perp} = \mathbf{0}$. Then $< \mathbf{w}, \alpha\mathbf{w} + \beta\mathbf{w}^{\perp} >= \alpha < \mathbf{w}, \mathbf{w} >= 0$ and $< \mathbf{w}^{\perp}, \alpha\mathbf{w} + \beta\mathbf{w}^{\perp} >= \beta < \mathbf{w}^{\perp}, \mathbf{w}^{\perp} >= 0$. Hence, $\alpha = \beta = 0$ as $< \mathbf{w}, \mathbf{w} >\neq 0$ and $< \mathbf{w}^{\perp}, \mathbf{w}^{\perp} >\neq 0$.

[4] (c) Suppose we are in the special case where $V = \mathbb{R}^n$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$. Show that $\dim W^{\perp} = n - \dim W$.

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for $W$. We note that $W^{\perp} = \text{Nul}A^T$ where $A$ is the matrix whose columns are $\mathbf{b}_1, \ldots, \mathbf{b}_n$. The rank of $A^T$ is the rank of $A$ which is the dimension of $W$. We know that $\text{rank}(A^T) + \dim \text{Nul}A^T = n$. Hence, $\dim W + \dim W^{\perp} = n$ as required.

[4] **5.** (a) Find the characteristic polynomial of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 2 \end{pmatrix}$.

We have that

$$\begin{aligned} p_A(\lambda) &= \begin{vmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 0 & 0 & 2-\lambda & 4 \\ 0 & 0 & 4 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)^2 \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)^2(4-4\lambda+\lambda^2-16) \\ &= (\lambda-1)^2(\lambda^2-4\lambda-12) \\ &= (\lambda-1)^2(\lambda-6)(\lambda+2) \end{aligned}$$

[4] (b) Determine a basis for the eigenspace of $A$ corresponding to the eigenvalue $1$.

We wish to determine a basis for Nul$(A-I)$.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A basis is given by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

[2] (c) Is $A$ orthogonally diagonalizable?

$A$ is not orthogonally diagonalizable as $A$ is not symmetric.

**6.** Let $\mathbf{b}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}1\\2\\0\\0\end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix}1\\2\\0\\0\end{pmatrix}, \mathbf{b}_4 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$.

[4] (a) Find an orthonormal set $\mathcal{C}$ such that Span $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\} =$ Span $\mathcal{C}$.

We note that Span $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\} =$ Span $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$. We apply Gram-Schmidt to get an orthogonal basis first.

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{b}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \\
\mathbf{v}_2 &= \mathbf{b}_2 - \frac{\mathbf{b}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
&= \begin{pmatrix}1\\2\\0\\0\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \\
&\sim \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} \\
\mathbf{v}_4 &= \mathbf{b}_4 - \frac{\mathbf{b}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{b}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&= \begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \\
&= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}.
\end{aligned}$$

Then $\left\{\frac{1}{\sqrt{2}}\mathbf{v}_1, \frac{1}{\sqrt{5}}\mathbf{v}_2, \mathbf{v}_4\right\}$ is an orthonormal basis for $W$.

[2] (b) What is the minimum distance between the vector $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ and a vector in Span $\{\mathbf{b}_1, \mathbf{b}_2\}$.

We note that Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for $W =$ Span $\{\mathbf{b}_1, \mathbf{b}_2\}$.

Then $\hat{\mathbf{y}} = \text{Proj}_W \mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Hence the minimum distance is $\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{5}$.

[4] **7.** (a) Suppose that $A$ is a square matrix such that $A^T A = I$. Prove that $\det A = \pm 1$.

Since $A^T A = I$, we have that $\det A^T \det A = 1$ and hence $(\det A)^2 = 1$ as $\det A = \det A^T$. Thus, $\det A = \pm 1$.

[2] (b) Let $\mathcal{C}$ be an orthonormal basis for $\mathbb{R}^3$. What is the volume of the parallelpiped spanned by the vectors in $\mathcal{C}$?

Let $A$ be the matrix whose columns are the vectors in $\mathcal{C}$. Then $A^T A = I$ as $\mathcal{C}$ is orthonormal. The volume of the associated parallelepiped is $|\det A| = 1$

**8.** Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for a vector space $V$.

[2] (a) What is the dimension of $V$?

The dimension of $V$ is $3$.

[2] (b) What is the maximum number of elements in a linearly-independent subset of $V$?

The maximum number of elements in a linearly-independent subset of $V$ is $3$.

[2] (c) Suppose $T : V \to V$ is a linear transformation such that $T(\mathbf{b}_1) = 2\mathbf{b}_1 + \mathbf{b}_2, T(\mathbf{b}_2) = \mathbf{b}_3, T(\mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_3$. Determine $[T]_{\mathcal{B}}$.

We have that

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

[4] (d) Suppose $T : \mathbb{P}_1 \to \mathbb{P}_2$ is the linear transformation given by $T(a_0 + a_1 t) = a_0 t + a_1 t^2$. Let $\mathcal{B}$ be the standard basis for $\mathbb{P}_1$ and $\mathcal{C}$ be the standard basis for $\mathbb{P}_2$. Determine $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$.

We have that

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$