Name:	**Student Number:**
(First) (Last), (CAPITAL letters please)	

MATH 232
FINAL EXAMINATION

Date: December 8, 2004 **Instructors:** J. Lester and G. Sabin
Time: 8:30 a.m. - 11:30 a.m.

INSTRUCTIONS

- **Do not lift the cover page until instructed !**
- **No aids or materials (including calculators or scrap paper) are allowed.**
- **For full credit show intermediate steps in your answers.**
- **All answers to be written directly on the question paper. Clearly strike out any work you do not want marked. If you run out of space use the back of the preceding page, but make sure you indicate that you have done so on the front page. Otherwise, any work on the back page will be considered rough work not part of your answer and will not be marked.**
- **The exam consists of ten questions and 16 pages (counting this one). The last page consists of the list of the vector space axioms.**

Question	Mark
1	/6
2	/12
3	/5
4	/12
5	/15
6	/5
7	/10
8	/8
9	/17
10	/10
Total	/100

MARKS

1. Let $\mathcal{S}$ be the plane with the equation, (6 pts)

$$2x + y - z = 6$$

Find the shortest distance from the point $P(3, 1, -1)$ to the plane $\mathcal{S}$ **and** the point Q on the plane $\mathcal{S}$ closest to P.

2.

(a) Let both P and Q be $n \times n$ matrices then solve the following equation for Q in terms of P, (2 pts)

$$(P + 3I)\,Q^T = \left[(Q + 3I)\,P^T\right]^T$$

(b) Given that (3 pts)

$$\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 5$$

find the value of

$$\det \begin{bmatrix} b + 2c & q + 2r & y + 2z \\ 2a & 2p & 2x \\ c & r & z \end{bmatrix}$$

(c) Find all values of k for which the following matrix is invertible, (2 pts)

$$B = \begin{bmatrix} 6 & 2 & 0 \\ -k^2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

Note it is not required to determine the inverse.

(d) Let E be an $n \times n$ matrix. Give five statements that are equivalent to the statement (5 pts)

E is invertible.

3. Let

$$A = \begin{bmatrix} 1 & 0 & -2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ -1 & -2 & -8 & 6 & -1 \\ 2 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 0 & 1 & -\frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is given that H and A are **row-equivalent**.

(a) What is the rank of A? (1 pt)

(b) Write down a basis for the row space of A (row(A)). (2 pts)

(c) Write down a basis for column space for A (col(A)) . (2 pts)

(d) Find a basis for the null space of A (null(A)). (4 pts)

(e) What is the nullity of A? (1 pt)

(f) Find a basis for the othogonal complement to row(A). (2 pts)

4. Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(a) Find A^{-1} (4 pts)

(b) Use A^{-1} found in (a) to solve the system $A\mathbf{x} = \mathbf{b}$ where (1 pt)

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

5.

(a) Find a basis for the subspace of $\mathbb{R}^3$ given by the plane, $3x - 2y + 5z = 0$. (3 pts)

(b) Let $\{\mathbf{u}, \mathbf{v}\}$ be a basis for a vector space V. Explain why or why not each of the following sets of vectors is a basis for V. (3 pts)

i. $\{\mathbf{0}, \mathbf{u} + \mathbf{v}\}$

ii. $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - 2\mathbf{v}\}$

iii. $\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$

(c) Show that $V = \{(x, y) \text{ in } \mathbb{R}^2 | x \leq 0, y \geq 0\}$ with the usual vector addition and scalar multiplication in $\mathbb{R}^2$, is **not** a vector space. (3 pts)

(d) Find another vector that can be added to $\{\mathbf{v}_1, \mathbf{v}_2\}$ to form a basis in $\mathbb{R}^3$ where (3 pts)

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

(e) Is (3 pts)

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \,\epsilon \mathbb{R} \right\}$$

a subspace of the vector space of all 2×2 matrices? Justify your answer.

6. Let $\mathcal{P}_1$ be the set of all polynomials of degree at most 1.

(a) Find a change of basis matrix , $P_{C \leftarrow B}$ where $B = \{1, x\}$ and $C = \{x, 1 + x\}$ in $\mathcal{P}_1$. (3 pts)

(b) Write the coordinate vector, $[p(x)]_C$ where $p(x) = 3 - x$ and the basis C is defined in (a). (2 pts)

7.

(a) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the a linear transformation defined by (3 pts)

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

Find $T\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(b) If $\mathcal{D}$ is the set of all real-valued differentiable functions show that $T : \mathcal{D} \longrightarrow \mathcal{D}$ defined by $T(f) = f' + f$ is a linear transformation. (3 pts)

(c) Let V and W be vector spaces of dimension n. **Prove** that a one-to-one linear transformation $S : V \longrightarrow W$ maps a basis for V to a basis for W. (4 pts)

8. Let $T : \mathcal{P}_2 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} 2a - b \\ a + b \\ -a \end{bmatrix}$$

(a) Find a basis for range(T). (2 pts)

(b) Find a basis for ker(T). (2 pts)

(c) What is the rank of T? (1 pt)

(d) State the definition of an isomorphism. (1 pt)

(e) Is T an isomorphism? Justify you answer. (2 pts)

9.

(a) Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

i. The eigenvalues of A are $\lambda = 1, 2, 2$. Use this fact to determine if A^{-1}exists. Justify your answer. (1 pt)

ii. Find the eigenspace for each distinct eigenvalue. (6 pts)

iii. Find a matrix P that diagonalizes A and the resulting diagonal matrix, D. Write an equation that relates A and D. (4 pts)

iv. Is A orthogonally diagonalizable? Why or why not? (2 pts)

(b) Prove that eigenvalues of similar matrices are the same. (4 pts)

10.

(a) The following vectors form a basis for a subspace W of $\mathbb{R}^3$: (3 pts)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Do $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthogonal basis for W? Justify your answer. If $\mathbf{v}_1$ and $\mathbf{v}_2$ do not form an orthogonal basis, use one of these vectors and another vector from W to form an orthogonal basis.

(b) Find an orthonormal basis for W in (a). (1 pt)

(c) Let (2 pts)

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Is A an orthogonal matrix? Justify your answer. If the matrix is orthogonal then find its inverse.

(d) Prove that an orthogonal set of k nonzero vectors, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$, in a subspace of $\mathbb{R}^n$ is always linearly independent. (4 pts)

Vector Space Rules

A set of vectors, V, with operations of addition and scalar multiplication defined on it, is a vector space if, for any vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in V and any scalar c, the following rules hold:

1. $\mathbf{u} + \mathbf{v}$ is a vector in V
2. $c\mathbf{u}$ is a vector in V
3. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
4. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
5. There is a vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u}$ in V.
6. For each vector $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $(cd)\mathbf{u} = c(d\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$