Math 232, Fall 2007
Final
Dec. 12, 2007

Last Name:	
First Name:	
SFU ID:	

1. DO NOT LIFT UP THE COVER PAGE UNTIL INSTRUCTED.
2. No calculators are allowed.
3. This test is comprised of 12 pages (including cover page)
4. Once the test begins, please check that all pages are intact.
5. Do ALL questions.
6. Clearly explain your answer. No credit will be given for just writing down the answer.
7. If the answer space provided is not sufficient, write your answer on the back of the previous page. Clearly mark the question number.
8. Good luck.

Question	Points	Score
1	8	
2	9	
3	12	
4	9	
5	9	
6	9	
7	11	
8	10	
9	12	
10	11	
Total:	100	

1. Let A be the matrix
$$\begin{pmatrix} 3 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 3 \end{pmatrix}.$$

   (a) (6 points) Compute the characteristic polynomial of A. Show all work. Your final answer should be of the form $ax^3 + bx^2 + cx + d$.

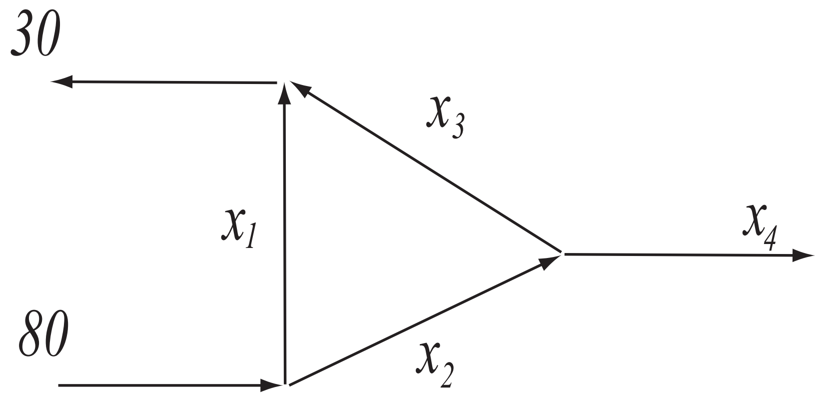
   (b) (2 points) Use part a. to decide whether A is invertible or not.

SOLUTION: a. We compute the determinant of the matrix $A - xI$:
$$\begin{pmatrix} 3-x & 0 & 2 \\ 1 & -x & 0 \\ 0 & -1 & 3-x \end{pmatrix}.$$

We use cofactor expansion along the first row to get
$$(3-x)(-x)(3-x) + 2(-1) = -x^3 + 6x^2 - 9x - 2.$$

b. If we plug in $x = 0$, we get the determinant of A, which is -2. Hence A is invertible.

2. (9 points) Assuming that all the flows are nonnegative in the figure below, what is the largest possible value for x_3? Show all work.



SOLUTION: Analyzing each intersection, we obtain the equations $x_1+x_2 = 80$, $x_1+x_3 = 30$, and $x_2 - x_3 - x_4 = 0$. We create the augmented matrix.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 80 \\ 1 & 0 & 1 & 0 & 30 \\ 0 & 1 & -1 & -1 & 0 \end{pmatrix}$$

Row reducing, we obtain the echelon form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 80 \\ 0 & -1 & 1 & 0 & -50 \\ 0 & 0 & 0 & -1 & -50 \end{pmatrix}$$

Notice x_3 is a free variable. Row 3 yields: $x_4 = 50$. Row 2 gives $x_2 = 50 + x_3$. Row 1 gives $x_1 = 30 - x_3$. Since x_1 cannot be negative, we see that $x_3 \leq 30$. Thus the maximum possible value of x_3 is 30.

3. True or False. Justify your answers.

   (a) (2 points) If an $n \times n$ matrix is diagonalizable it has n distinct eigenvalues.

   (b) (2 points) If A is a square matrix and $A\vec{v}$ is in the span of $\vec{v}$, then $\vec{v}$ is an eigenvector of A.

   (c) (2 points) If an $n \times n$ matrix is upper-triangular then its eigenvalues are just the entries along its main diagonal.

   (d) (2 points) If A is a 3×3 matrix and $A^2 = 0$ then $A = 0$.

   (e) (2 points) If a system of equations has at least 2 solutions then it has at least three solutions.

   (f) (2 points) If $\det(A) = 3$ and $\det(B) = 6$ then $\det(A^{-1}B) = 2$.

SOLUTIONS:

a. False, consider the 2×2 identity matrix.

b. False, if $\vec{v}$ is the zero vector this works and yet it can never be an eigenvector.

c. True, since the determinant of an upper-triangular matrix is just the product of the diagonal terms, the characteristic polynomial of A is just the product of things of the form $(x-a)$, where a is an entry on the main diagonal of A.

d. False, consider the matrix that has zeros everywhere except for in the entry that is in the first row and second column.

e. True. A system of equations can have either 0, 1, or infinitely many solutions. In particular, if it has at least two solutions, it has infinitely many solutions.

f. True. Since $\det(A^{-1}) = 1/3$ and $\det(A^{-1}B) = \det(A^{-1})\det(B)$.

4. Which of the following sets are subspaces of $\mathbb{R}^4$? Justify your answer.

   (a) (3 points)

$$W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \; : \; x + 3y = 1 \text{ and } x + 2z = 3 \right\}$$

   (b) (3 points)

$$W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \; : \; x + 3y = 0 \text{ and } x + 2z = 0 \right\}$$

   (c) (3 points)

$$W_3 = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \; : \; x + 3y = 0 \text{ or } x + 2z = 0 \right\}$$

SOLUTION:

a. It is not a subspace. Notice the zero vector is not in this space.

b. Yes, this is a subspace. This is the null space of the matrix

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$

c. No, this is not a subspace. Notice it is not closed under addition. Since

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in W_3$$

and

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in W_3$$

but their sum:

$$\begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix} \notin W_3$$

5. In the table below, data are given for two variables x and y.

x	y
-2	-3
-1	-1
0	1
1	3
2	4

(a) (7 points) Compute the line of best fit for these data. Show all work.

(b) (2 points) What does the line of best fit estimate the value of y will be when $x = 5$.

SOLUTION: We let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \\ 4 \end{bmatrix}.$$

Then the line of best fit is $y = C_0 + C_1 x$, where

$$\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$$

is the least squares solution to the equation

$$A\vec{x} = \vec{b}.$$

Thus we compute

$$(A^T A)^{-1} A^T \vec{b}$$

to find this solution. Solving, we get $C_0 = 4/5$ and $C_1 = 9/5$, so the line of best fit is

$$y = 4/5 + 9x/5.$$

Notice that when $x = 5$, $y = 49/5$, hence the line of best fit estimates y to be $49/5$.

6. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 + 3x_1 \\ x_1 + x_2 \end{bmatrix}$$

and

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

(a) (7 points) Compute the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$, $[T]_{\mathcal{B}}^{\mathcal{C}}$. Show all work.

(b) (2 points) if

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

what is $[T(\vec{x})]_{\mathcal{C}}$?

SOLUTION: We first compute what T does to the basis vectors in $\mathcal{B}$.

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We now find the coordinates of these vectors with respect to the basis $\mathcal{C}$. Looking at the first coordinates we see:

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} -3 & -2 \\ 5 & 3 \end{pmatrix}.$$

For part b., notice that

$$[T(\vec{x})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[\vec{x}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{C}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

7. (a) (8 points) Find $\mathrm{Col}(A)^{\perp}$ when

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 \\ 1 & 2 & 3 & 6 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \end{pmatrix}.$$

Show all work.

(b) (3 points) Use your work in part a. to determine the dimension of $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$.

SOLUTION: We recall that $\mathrm{Col}(A)^{\perp} = \mathrm{Nul}(A^T)$ Since

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 3 & 3 & 1 & 1 \\ 5 & 6 & 3 & 4 \end{pmatrix},$$

we can compute the Null space by row reducing the augmented matrix to the matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Notice if we regard this as the augmented matrix of a system of equations in variables $x_1, \ldots, x_4$, then x_4 is free, $x_3 = -x_4$, $x_2 = -x_4$ and $x_1 = x_4$. Thus the null space is just the span of the vector:

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

By the properties of orthogonal complements, the Column space is 3 dimensional. The Null space is then 1 dimensional, since the sum of the dimensions of the null and column space is the number of columns, which in this case is 4.

8. (a) (8 points) Find the orthogonal projection of

$$\vec{v} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}$$

onto the Column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Show all work.

(b) (2 points) Find $\vec{y} \in \mathrm{Col}(A)$ and $\vec{z} \in \mathrm{Col}(A)^{\perp}$ such that $\vec{v} = \vec{y} + \vec{z}$.

SOLUTION: By row reduction, we see that A is row equivalent to

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the first two columns are pivot columns. It follows that the first two columns of A form a basis for the column space of A. We use Gram-Schmidt to compute an orthogonal basis:

$$\vec{u_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\vec{u_2} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus $\{\vec{u_1}, \vec{u_2}\}$ is an orthogonal basis for the column space. We now use the orthogonal projection formula:

$$\widehat{v} = \frac{\vec{v} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}} \vec{u_1} + \frac{\vec{v} \cdot \vec{u_2}}{\vec{u_2} \cdot \vec{u_2}} \vec{u_2} = 2\vec{u_1} + \vec{u_2} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

For part b, notice that $\vec{v} = \widehat{v} + \vec{z}$, where $\vec{z} \in \mathrm{Col}(A)^{\perp}$ and $\widehat{v} \in \mathrm{Col}(A)$. Thus

$$\vec{z} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

and $\vec{y} = \widehat{v}$.

9. (a) (8 points) Show that $2h+3$ is an eigenvalue of the matrix

$$\begin{pmatrix} 2h+2 & 1 & 0 \\ h+3 & h & 1 \\ 2-h & h-2 & 2 \end{pmatrix}.$$

(b) (4 points) Find an eigenvector corresponding to the eigenvalue $2h+3$. Show all work.

SOLUTION: We have to show that $A-(2h+3)I$ is not invertible. To do this, we compute the determinant of $A-(2h+3)I$:

$$\begin{pmatrix} -1 & 1 & 0 \\ h+3 & -h-3 & 1 \\ 2-h & h-2 & -1-2h \end{pmatrix}.$$

We use cofactor expansion along the third row:

$$-1((2-h)-(2-h))+(-1-2h)((h+3)-(h+3))=0$$

To find an eigenvector we must find an element in the Null space of $A-(2h+3)I$. To do this we row reduce the augmented matrix to the matrix:

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice if we regard this as the augmented matrix of a system of equations in variables $x_1,\ldots,x_3$, then x_2 is free, $x_3=0$, $x_1=x_2$. Taking $x_2=1$, we get an eigenvector:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

10. Let

$$A = \begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}.$$

(a) (6 points) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$. Show all work.

(b) (5 points) Find a formula for A^{100}. (Hint: Use part a. Your formula should be given by a 2×2 matrix whose entries are of the form $C_0 a^{100} + C_1 b^{100}$, where C_0, C_1, a, b are constants.) Show all work.

SOLUTION: We first compute the characteristic polynomial of A, which is $\det(A - xI) = x^2 - 3x + 2 = (x-2)(x-1)$. Thus the eigenvalues are 2 and 1. We next compute eigenvectors corresponding to each eigenvalue:
$c = 2$:

$$A - 2I = \begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix}.$$

Row reducing the augmented matrix, we find the vector

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

is an eigenvector.
$c = 1$:

$$A - I = \begin{pmatrix} 10 & -15 \\ 6 & -9 \end{pmatrix}.$$

Row reducing the augmented matrix, we find the vector

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

is an eigenvector. If we take S to be a matrix whose columns are the eigenvectors,

$$S = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},$$

then $S^{-1}AS = D$, where D is the diagonal matrix

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $A = SDS^{-1}$. For part b. Note that $A^{100} = SD^{100}S^{-1}$. Hence

$$A^{100} = SD^{100}S^{-1} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 5 \cdot 2^{100} & 3 \\ 3 \cdot 2^{100} & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$$

This is equal to

$$\begin{pmatrix} 10 \cdot 2^{100} - 9 & -15 \cdot 2^{100} + 15 \\ 6 \cdot 2^{100} - 6 & -9 \cdot 2^{100} + 10 \end{pmatrix}.$$