Simon Fraser University

Math 232 Final Exam Solution

1. T F T T F F F F T T T T

2.

a. $\left(\begin{array}{ccc|c} 1 & 3 & 9 & -2 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$

b. $\left(\begin{array}{ccc|c} 1 & 3 & 9 & -2 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \underset{\substack{r_1 \leftrightarrow r_2 \\ r_2 \leftrightarrow r_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 3 & 9 & -2 \end{array}\right) \underset{\substack{r_3 \leftarrow r_3 - r_1 \\ r_3 \leftarrow r_3 - 3r_2}}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Then if $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ we have that $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ for $s \in \mathbb{R}$.

3.

a. $\det B \underset{\substack{r_1 \leftrightarrow r_2 \\ r_3 \leftarrow r_3 + 2r_1}}{=} -\det A = -7$

b. $\det B \neq 0$ so B is invertible.

4.

a. $x_i = \dfrac{\det\left(A_i\left(\vec{b}\right)\right)}{\det A}$

b. $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$, so we have

$\det\left(A_3\left(\vec{b}\right)\right) = \det\begin{pmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix} = 0 + 6 - 4 - 0 - 2 - 4 = -4$,

$\det A = 0 + 6 - 1 - 0 + 3 - 4 = 4$ and

$x_3 = \dfrac{\det\left(A_3\left(\vec{b}\right)\right)}{\det A} = \dfrac{-4}{4} = -1$

5. $\{1, t, t^2\}$ is a basis for $\mathbb{P}_2$.

$$p_0 = 1$$
$$p_1 = t - \frac{\langle t, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 = t - \frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle} 1 = t - \frac{-1+0+1}{1+1+1} 1 = t - 0 = t$$
$$p_2 = t^2 - \frac{\langle t^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1 - \frac{\langle t^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 = t^2 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1$$
$$= t^2 - \frac{-1+0+1}{1+1+1} t - \frac{1+0+1}{1+1+1} 1 = t^2 - 0t - \tfrac{2}{3} = t^2 - \tfrac{2}{3}$$

So we have that $\{1, t, t^2 - \frac{2}{3}\}$ is an orthogonal basis for $\mathbb{P}_2$.

6. $$A \underset{\substack{r_1 \leftarrow -r_1 \\ r_2 \leftarrow r_2 - 2r_1 \\ r_3 \leftarrow r_3 + 2r_1}}{\sim} \begin{pmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -3 & 3 \end{pmatrix} \underset{\substack{r_2 \leftarrow \frac{1}{2} r_2 \\ r_3 \leftarrow r_3 + 3r_2}}{\sim} \begin{pmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{r_1 \leftarrow r_1 - 2r_2}{\sim} \begin{pmatrix} 1 & -2 & 0 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the reduced echelon form of A we see that $\mathcal{B}_C = \left\{ \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \\ 7 \end{pmatrix} \right\}$, $\mathcal{B}_N = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$, and

$\mathcal{B}_R = \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ are bases for Col$A$, Nul$A$, and Row$A$, respectively.

7.

a. Let $[A]_{\mathcal{B}} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{pmatrix} 2 & 2 \\ -6 & 2 \end{pmatrix} = x \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This gives a system of linear equations whose augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 2 \\ -1 & -1 & -6 \\ 1 & 0 & 2 \end{array}\right) \underset{\substack{r_2 \leftarrow r_2 + r_1 \\ r_3 \leftarrow r_3 + r_1 \\ r_4 \leftarrow r_4 - r_1}}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{array}\right) \underset{r_3 \leftarrow r_3 + r_2}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus $[A]_{\mathcal{B}} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

b. $P_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{\mathcal{C}} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\mathcal{C}} \right]$ so we must write the elements of $\mathcal{B}$ as a linear combination of the basis $\mathcal{C}$. Let $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{\mathcal{C}} = \begin{pmatrix} w \\ x \end{pmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\mathcal{C}} = \begin{pmatrix} y \\ z \end{pmatrix}$, then

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = w\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} + x\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = y\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

The above correspond to two systems of linear equations, which we can solve simultaneously

$$\left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 2 & 0 & -1 & 1 \\ 0 & -2 & -1 & -1 \\ -1 & 1 & 1 & 0 \end{array}\right) \underset{\substack{r_1 \leftrightarrow r_2 \\ r_2 \leftrightarrow r_3 \\ r_4 \leftarrow r_4 - r_3}}{\sim} \left(\begin{array}{cc|cc} 2 & 0 & -1 & 1 \\ 0 & -2 & -1 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \underset{\substack{r_1 \leftarrow \frac{1}{2}r_1 \\ r_2 \leftarrow -\frac{1}{2}r_2 \\ r_3 \leftarrow r_3 + r_1 \\ r_3 \leftarrow r_3 - r_2}}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{\mathcal{C}} = \begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\mathcal{C}} = \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, and so $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

c. $[A]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

8.

a. $A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$

$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

Then the normal equations are $\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}\vec{x} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

b. $\left(\begin{array}{cc|c} 3 & -3 & 6 \\ -3 & 3 & -6 \end{array}\right) \underset{\substack{r_1 \leftarrow \frac{1}{3}r_1 \\ r_2 \leftarrow r_2 + 3r_1}}{\sim} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right)$

Then $\vec{x}' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (for any $s \in \mathbb{R}$) are the least-squares solutions of $A\vec{x} = \vec{b}$.

c. $A\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2+s \\ s \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

Then

$$\left\|\vec{b} - A\vec{x}'\right\| = \left\|\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}\right\| = \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{14}$$

is the least-squares error.

9.

a. A transformation $T : V \to W$ is said to be **linear** if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$T(c\vec{u}) = cT(\vec{u})$$

for all $\vec{u}, \vec{v} \in V$ and all scalars c.

b. Let $p, q \in \mathbb{P}_2$, then

$$T\left(p(t) + q(t)\right) = (1-t)\cdot\left(p(t) + q(t)\right) = (1-t)\cdot p(t) + (1-t)q(t) = T\left(p(t)\right) + T\left(q(t)\right)$$

$$T\left(c \cdot p(t)\right) = (1-t)\cdot\left(c \cdot p(t)\right) = c\cdot(1-t)\cdot p(t) = c\cdot T\left(p(t)\right)$$

And so T is linear.

c. $M = \left[\left[T(1)\right]_{\mathcal{C}} \quad \left[T(t)\right]_{\mathcal{C}} \quad \left[T\left(t^2\right)\right]_{\mathcal{C}}\right] = \left[[1-t]_{\mathcal{C}} \quad \left[t - t^2\right]_{\mathcal{C}} \quad \left[t^2 - t^3\right]_{\mathcal{C}}\right] = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

is the matrix of T relative to $\mathcal{B}$ and $\mathcal{C}$.

10.

a. A is orthogonally diagonalisable if there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

b. A is symmetric.

c. $$f_A(\lambda)=\det(A-\lambda I)=\det\begin{pmatrix}1-\lambda & 5\\ 5 & 1-\lambda\end{pmatrix}=(1-\lambda)^2-5^2=(1-\lambda-5)(1-\lambda+5)$$
$$=(-4-\lambda)(6+\lambda)$$

The eigen-values of A are –4 and 6.

$\lambda=-4$: $A+4I=\begin{pmatrix}5 & 5\\ 5 & 5\end{pmatrix}\sim\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$ so that $\vec{v}_{-4}=\begin{pmatrix}-1\\ 1\end{pmatrix}$ is an eigen-vector of A.

$\lambda=6$: $A-6I=\begin{pmatrix}-5 & 5\\ 5 & -5\end{pmatrix}\sim\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$ so that $\vec{v}_6=\begin{pmatrix}1\\ 1\end{pmatrix}$ is an eigen-vector of A.

We normalize $\vec{v}_{-4}$ and $\vec{v}_6$ as the columns of P need to be unit vectors:

$$\vec{u}_{-4}=\frac{1}{\|\vec{v}_{-4}\|}\vec{v}_{-4}=\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\vec{u}_6=\frac{1}{\|\vec{v}_6\|}\vec{v}_6=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

Then let $P=\begin{pmatrix}\vec{u}_{-4} & \vec{u}_6\end{pmatrix}=\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ and $D=\begin{pmatrix}-4 & 0\\ 0 & 6\end{pmatrix}$, then $A=PDP^T$.

d. $$A=-4\vec{u}_{-4}\vec{u}_{-4}{}^T+6\vec{u}_6\vec{u}_6{}^T=-4\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}+6\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=-4\begin{pmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{1}{2}\end{pmatrix}+6\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

11.

a. S is linearly independent if whenever we choose scalars $c_1,\ldots,c_p$ such that $c_1\vec{v}_1+\ldots+c_p\vec{v}_p=\vec{0}$, we must have that $c_1=\ldots=c_p=0$.

b. Let $T:V\to W$ be a linear transformation of vector spaces V and W, and suppose that $\{\vec{v}_1,\ldots,\vec{v}_p\}\subseteq V$. If the set $\{T(\vec{v}_1),\ldots,T(\vec{v}_p)\}$ is linearly independent, prove that the set $\{\vec{v}_1,\ldots,\vec{v}_p\}$ is also linearly independent.

Suppose that $\{\vec{v}_1,\ldots,\vec{v}_p\}$ is linearly dependent. Then there are scalars $c_1,\ldots,c_p$, not all zero such that $c_1\vec{v}_1+\ldots+c_p\vec{v}_p=\vec{0}$. Then

$$\vec{0}=T(\vec{0})=T(c_1\vec{v}_1+\ldots+c_p\vec{v}_p)=c_1T(\vec{v}_1)+\ldots+c_pT(\vec{v}_p)$$

and we have constructed a non-trivial linear combination of $T(\vec{v}_1),\ldots,T(\vec{v}_p)$ equaling zero.

Thus $\left\{T\left(\vec{v}_1\right),\ldots,T\left(\vec{v}_p\right)\right\}$ is a linearly dependent set.

12.

a. A set W to be a **subspace** of V if (i) $W \subseteq V$, (ii) $W \neq \varnothing$, (iii) W is closed under vector addition, and (iv) W is closed under scalar multiplication.

b. Let W be a subspace of $\mathbb{R}^n$. Show that the set $W^\perp$, the orthogonal complement of W, is also a subspace of $\mathbb{R}^n$.

(i) $W^\perp \subseteq \mathbb{R}^n$ follows from the fact that $W \subseteq \mathbb{R}^n$.
(ii) For any $\vec{w} \in W$, $\vec{w} \cdot \vec{0} = 0$. This means that $\vec{0} \in W^\perp$, and so $W^\perp \neq \varnothing$.
(iii) Choose $\vec{y}, \vec{z} \in W^\perp$. Then for any $\vec{w} \in W$, $\vec{w} \cdot \left(\vec{y} + \vec{z}\right) = \vec{w} \cdot \vec{y} + \vec{w} \cdot \vec{z} = 0 + 0 = 0$ and so $\vec{y} + \vec{z} \in W^\perp$.
(iv) Finally, choose $\vec{z} \in W^\perp$ and any scalar c. For any $\vec{w} \in W$, $\vec{w} \cdot \left(c\vec{z}\right) = c\left(\vec{w} \cdot \vec{z}\right) = c0 = 0$ and so $c\vec{z} \in W^\perp$.

Together, (i) – (iv) show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

13.

a. A and B are said to be similar if there is some invertible matrix P such that $A = PBP^{-1}$.

b. Let A and B be similar matrices and choose P such that $A = PBP^{-1}$. Then

$$\begin{aligned} f_A(\lambda) &= \det(A - \lambda I) = \det\left(PBP^{-1} - \lambda PP^{-1}\right) = \det\left(P(B - \lambda I)P^{-1}\right) \\ &= \det(P)\det(B - \lambda I)\det\left(P^{-1}\right) = \det(P)\det\left(P^{-1}\right)\det(B - \lambda I) \\ &= \det\left(PP^{-1}\right)\det(B - \lambda I) = \det(I)\det(B - \lambda I) = \det(B - \lambda I) = f_B(\lambda) \end{aligned}$$

and A and B have the same characteristic equation.