Simon Fraser University

Math 232

Final Exam
Instructor : Aaron Bradford

Date: 11 August 2007
Time: 8:30 – 11:30

Last Name (print): ________________________ **First Name: ________________________**

Signature: ________________________________ **SFU Email ID: ______________________**

Instructions:

1. DO NOT OPEN THIS EXAM UNTIL INSTRUCTED TO DO SO.
2. Ensure that you have 13 pages of questions.
3. No calculators, notes or books are allowed.
4. Except for question 1, credit will not be given for answers with no explanation.
5. Answer each question in the space provided. Continue on the back of the previous page if necessary.
6. Have your picture ID ready for inspection.
7. You will have 180 minutes to complete the exam. If you are caught writing after this time limit, you will be assessed a 5 point penalty.
8. With 10 minutes remaining in the exam, you will be asked to stay seated until time is up.
9. Good luck!

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	**Total**
Mark														
Maximum	6	4	3	4	4	5	7	7	5	9	4	6	4	**68**

1. (**½ point each**) Mark the following statements as either true or false. No explanation is required.

a. ____ Whenever a system has free variables, the solution set contains many solutions.

b. ____ If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

c. ____ A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

d. ____ A vector space is also a subspace.

e. ____ A linearly independent set in a subspace *H* is a basis for *H*.

f. ____ Row operations preserve the linear dependence relations among the rows of *A*.

g. ____ If $\lambda + 5$ is a factor of the characteristic polynomial of *A*, then 5 is an eigen-value of *A*.

h. ____ If *A* is invertible, then *A* is diagonalisable.

i. ____ If *A* is diagonalisable and *B* is similar to *A*, then *B* is also diagonalisable.

j. ____ Every orthogonal set is linearly independent.

k. ____ If *W* is a subspace of $\mathbb{R}^n$ and if $\vec{v}$ is in both *W* and $W^\perp$, then $\vec{v}$ must be the zero vector.

l. ____ If $A = QR$ is a QR-Factorisation of *A*, then the columns of *Q* form an orthonormal basis for $\text{Col}A$.

2. (**1 point – 3 points**) Suppose that you are given the following system of linear equations

$$\begin{cases} x_1 + 3x_2 + 9x_3 = -2 \\ x_1 + 3x_3 = 1 \\ x_2 + 2x_3 = -1 \end{cases}$$

a. What is the augmented matrix of this system?

b. Find the parametric vector form of the general solution for the system.

3. (**2 points – 1 point**) Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $B = \begin{pmatrix} d & e & f \\ a & b & c \\ g-2a & h-2b & i-2c \end{pmatrix}$.

a. Given that $\det A = 7$, what is $\det B$?

b. Is B invertible? Justify.

4. (**1 point – 3 points**) The following system of linear equations has a unique solution:

$$\begin{cases} 2x_1 + x_2 + x_3 = 4 \\ -x_1 + 2x_3 = 2 \\ 3x_1 + x_2 + 3x_3 = -2 \end{cases}$$

a. Cramer's Rule gives a formula for x_i, i^{th} entry in the solution vector $\bar{x}$. What is this formula?

b. Use Cramer's Rule to calculate the value of x_3.

5. (**4 points**) Let $\mathbb{P}_2$ be the inner product space whose inner product is defined as

$$\langle p(t), q(t) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

for any polynomials p and q in $\mathbb{P}_2$. Find an orthogonal basis for $\mathbb{P}_2$.

6. (**5 points**) Let $A = \begin{pmatrix} -1 & 2 & -2 & -4 \\ 2 & -4 & 6 & 6 \\ -2 & 4 & -7 & -5 \end{pmatrix}$. Find a basis for each of Col$A$, Nul$A$, and Row$A$.

7. (**3 points – 3 points – 1 point**) $\mathcal{B}=\left\{\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}$ and $\mathcal{C}=\left\{\begin{pmatrix}-1 & 2\\ 0 & -1\end{pmatrix},\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\right\}$ are two bases for a subspace W of $M_{2\times 2}$, the vector space of 2×2 matrices.

a. If $A=\begin{pmatrix}2 & 2\\ -6 & 2\end{pmatrix}\in W$ then find $[A]_{\mathcal{B}}$, the $\mathcal{B}$-co-ordinate vector of A.

b. Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$, the change of co-ordinates matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$.

c. Calculate $[A]_{\mathcal{C}}$, the $\mathcal{C}$-co-ordinate vector of A.

8. (**2 points – 3 points – 2 points**) Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$.

a. Give the matrix form of the normal equations.

b. Find all least-squares solutions of $A\vec{x} = \vec{b}$.

c. Compute the least-squares error associated with the least-squares solutions found in (b).

9. **(1 point – 2 points – 2 points)**

a. For vector spaces, V and W, define what it means for a transformation $T: V \to W$ to be **linear**.

b. Show that the transformation $T: \mathbb{P}_2 \to \mathbb{P}_3$ given by $p(t) \mapsto (1-t) \cdot p(t)$ is linear by verifying that it satisfies the definition of linear.

c. Find the matrix of the transformation T relative to the bases $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{1, t, t^2, t^3\}$.

10. (**1 point – 1 point – 5 points – 2 points**)

a. Define what it means for a matrix A to be **orthogonally diagonalisable**.

b. Let $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$. Without calculation, how do we know that A is orthogonally diagonalisable?

c. Find an orthogonal diagonalisation of A.

d. Using your work from (c), find a spectral decomposition of A.

11. (**1 point – 3 points**)

a. Define what it means for the set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ to be a **linearly independent** subset of *V*.

b. Let $T : V \to W$ be a linear transformation of vector spaces *V* and *W*, and suppose that $\{\vec{v}_1, \ldots, \vec{v}_p\} \subseteq V$. If the set $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is linearly independent, prove that the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is also linearly independent.

12. (**2 points – 4 points**)

a. Define what it means for a set *W* to be a **subspace** of *V*.

b. Let *W* be a subspace of $\mathbb{R}^n$. Show that the set $W^{\perp}$, the orthogonal complement of *W*, is also a subspace of $\mathbb{R}^n$.

13. (**1 point – 3 points**)

a. Define what we mean when we say that two $n \times n$ matrices, A and B, are **similar**.

b. Show that if two $n \times n$ matrices are similar, then they share the same characteristic polynomial.