Math 232 Final Examination Solution April 10, 2007

1. Consider the matrix
$$\begin{pmatrix} 2 & 8 & -2 \\ 3 & 13 & -4 \\ -1 & -7 & 3 \end{pmatrix}$$

   (a) (3 points) Compute $\det(A)$.

Answer

We use row reduction to compute the determinant:

$$\det\begin{pmatrix} 2 & 8 & -2 \\ 3 & 13 & -4 \\ -1 & -7 & 3 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 4 & -1 \\ 3 & 13 & -4 \\ -1 & -7 & 3 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \end{pmatrix} =$$

$$2\det\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = -2$$

   (b) (3 points) Compute A^{-1}.

Answer

We use row reduction on an augmented matrix. We join the computations after the first two steps made above:

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -3 & 2 & \frac{1}{2} & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 4 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -1 & -4 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{9}{2} & -3 & -1 \\ 0 & 1 & 0 & \frac{5}{2} & -2 & -1 \\ 0 & 0 & 1 & 4 & -3 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{11}{2} & 5 & 3 \\ 0 & 1 & 0 & \frac{5}{2} & -2 & -1 \\ 0 & 0 & 1 & 4 & -3 & -1 \end{array}\right), \text{ hence } A^{-1} = \begin{pmatrix} -\frac{11}{2} & 5 & 3 \\ \frac{5}{2} & -2 & -1 \\ 4 & -3 & -1 \end{pmatrix}$$

   (c) (3 points) Solve the system

$$\left\{\begin{array}{rrrcr} 2x_1 & +8x_2 & -2x_3 & = & -2 \\ 3x_1 & +13x_2 & -4x_3 & = & -5 \\ -1x_1 & -7x_2 & +3x_3 & = & 5 \end{array}\right.$$

Answer

With A^{-1} this is really easy:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = A^{-1}\begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{11}{2} & 5 & 3 \\ \frac{5}{2} & -2 & -1 \\ 4 & -3 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

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2. Consider the matrix

$$A = \begin{pmatrix} 10 & -12 \\ 6 & -7 \end{pmatrix}$$

(a) (3 points) Compute the characteristic polynomial of A.

Answer

$$\chi_A(\lambda) = \det \begin{pmatrix} 10-\lambda & 12 \\ 6 & -7-\lambda \end{pmatrix} = (10-\lambda)(-7-\lambda) + 72 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

(b) (4 points) Compute the eigenvalues of A and the corresponding eigenspaces.

Answer

The eigenvalues are the roots of the characteristic polynomial, i.e., $1, 2$.
The eigenspace for $\lambda = 1$ is:

$$\text{Nul} \begin{pmatrix} 9 & -12 \\ 6 & -8 \end{pmatrix} = \text{Nul} \begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix} = \text{Span}\{\begin{pmatrix} 4 \\ 3 \end{pmatrix}\}$$

The eigenspace for $\lambda = 2$ is:

$$\text{Nul} \begin{pmatrix} 8 & -12 \\ 6 & -9 \end{pmatrix} = \text{Nul} \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} = \text{Span}\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\}$$

(c) (2 points) Give an invertible matrix P and a diagonal matrix D such that $A = P^{-1}DP$.

Answer

Since the columns of $P = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ are eigenvectors of A for the eigenvalues $1, 2$ respectively, we have that for $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ we have $A = P^{-1}DP$.

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3. Let $W \subset \mathbb{R}^3$ be the subspace spanned by $\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \}$.

   (a) (3 points) Compute a basis for $W^\perp$. Explain your method.

Answer

We need a vector that is orthogonal to both given vectors. Using that $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$, we need:

$$\mathbf{w} \in \operatorname{Nul} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \operatorname{Nul} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & -1 \end{pmatrix} = \operatorname{Span}\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \}$$

   (b) (3 points) Compute the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto W and onto $W^\perp$. Explain your method.

Answer

The projection on the 1-dimensional space $W^\perp$ is straightforward:

$$\operatorname{proj}_{W^\perp} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{3} \end{pmatrix}$$

Using that $\operatorname{proj}_W(\mathbf{v}) + \operatorname{proj}_{W^\perp}(\mathbf{v}) = \mathbf{v}$, we compute

$$\operatorname{proj}_W \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \operatorname{proj}_{W^\perp} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ -\frac{1}{3} \end{pmatrix}$$

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4. Consider the following basis for $\mathbb{R}^3$: $\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}3\\-1\\-1\end{pmatrix} \}$.

   (a) (2 points) Give the definition of an orthogonal basis.

Answer

An orthogonal basis of a vector space $W \subset \mathbb{R}^n$ is a set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_r\}$ such that

- $\mathcal{B}$ spans W, i.e., $W = \text{Span} = \{\mathbf{b}_1, \ldots, \mathbf{b}_r\}$,
- $\mathcal{B}$ is linearly independent
- The elements of $\mathcal{B}$ are mutually orthogonal, i.e., $\mathbf{b}_i \cdot \mathbf{b}_j = 0$ for all $i \neq j$.

In fact, if you know that none of the vectors in $\mathcal{B}$ is the zero vector and that they are mutually orthogonal then the linear independence of $\mathcal{B}$ follows.

   (b) (2 points) Show that the given basis for $\mathbb{R}^3$ is not an orthogonal basis.

Answer

The vectors are not mutually orthogonal:

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}2\\0\\1\end{pmatrix} = 2 \neq 0$$

(c) (4 points) Perform the Gram-Schmidt orthogonalisation process on the given basis to obtain an orthogonal basis for $\mathbb{R}^3$. You may rescale vectors if it makes the arithmetic easier for you.

Answer

Given the vectors $\mathbf{x}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix}2\\0\\1\end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix}3\\-1\\-1\end{pmatrix}$ we perform the Gram-Schmidt process to produce an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}\ ;\ \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix}}{\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix}} \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{pmatrix}3\\-1\\-1\end{pmatrix} - \frac{\begin{pmatrix}3\\-1\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix}}{\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix}} \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{\begin{pmatrix}3\\-1\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix}} \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix}3\\-1\\-1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{3}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

(d) (2 points) Give the orthogonal projection of $\begin{pmatrix}3\\-1\\-1\end{pmatrix}$ onto the orthogonal complement of $\text{Span}\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\}$. Explain your answer.

Answer

From part (c) we know that the projection of $\mathbf{x}_3$ onto $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\mathbf{v}_1, \mathbf{v}_2$ is $\mathbf{x}_3 - \mathbf{v}_3$, so the projection onto the orthogonal complement is

$$\mathbf{v}_3 = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

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5. An experiment has yielded the following measurements for the quantities (x, y): $(0, 1), (1, 3), (2, 3), (3, 1)$. Some theory predicts that $y = \beta_0 + x\beta_1$.

   (a) (2 points) Express the problem of determining β_0, β_1 as a system of linear equations. Is the system consistent? Explain why (not).

Answer

For each measurement pair, we get an equation for β_0, β_1:

$$\begin{cases} \beta_0 + 0\beta_1 = 1 \\ \beta_0 + 1\beta_1 = 3 \\ \beta_0 + 2\beta_1 = 3 \\ \beta_0 + 3\beta_1 = 1 \end{cases} \text{i.e.,} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

   (b) (2 points) For a system of linear equations $A\mathbf{x} = \mathbf{b}$, describe the *least squares solution* to this system in terms of A, $\mathbf{b}$ and orthogonal projections.

Answer

The *least squares solution* to a system $A\mathbf{x} = \mathbf{b}$ is a solution to the system $A\mathbf{x} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto the column space of A. The vector $\hat{\mathbf{b}}$ is the closest vector to $\mathbf{b}$ for which the system is consistent.

   (c) (4 points) Compute the least squares line for the points given.

Answer

We can determine the least squares solution to a system $A\mathbf{x} = \mathbf{b}$ by solving $A^T A\mathbf{x} = A^T\mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

i.e.,

$$\left(\begin{array}{cc|c} 4 & 6 & 8 \\ 6 & 14 & 12 \end{array} \right) \sim \left(\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 5 & 0 \end{array} \right)$$

so $\beta_0 = 2$ and $\beta_1 = 0$. The least squares line for these points is $y = 2 + 0x$.

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6. (3 points) Let A be an $n \times m$ matrix. Prove that if $\mathbf{v} \in \mathrm{Nul}(A)$ then $\mathbf{v}$ is orthogonal to all vectors in the row space of A.

Answer

Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be the vectors that make up the rows of A, i.e., $A = \begin{pmatrix} - & \mathbf{a}_1^T & - \\ & \vdots & \\ - & \mathbf{a}_n^T & - \end{pmatrix}$. If $\mathbf{v} \in \mathrm{Nul}(A)$ then

$$\mathbf{0} = A\mathbf{v} = \begin{pmatrix} - & \mathbf{a}_1^T & - \\ & \vdots & \\ - & \mathbf{a}_n^T & - \end{pmatrix} \mathbf{v} = \begin{pmatrix} \mathbf{a}_1^T\mathbf{v} \\ \vdots \\ \mathbf{a}_n^T\mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{v} \\ \vdots \\ \mathbf{a}_n \cdot \mathbf{v} \end{pmatrix}$$

so, we see that $\mathbf{v}$ is orthogonal to each $\mathbf{a}_i$. But then, if $\mathbf{w}$ is in the row space of A, then $\mathbf{w} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$ and therefore,

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot (c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n) = c_1\mathbf{v} \cdot \mathbf{a}_1 + \cdots + c_n\mathbf{v} \cdot \mathbf{a}_n = 0$$

so indeed, $\mathbf{v}$ is orthogonal to any vector in $\mathrm{Row}(A)$

7. We consider the vector space $\mathbb{R}^{3\times 3}$ of 3×3 matrices. We consider the subset of symmetric matrices:

$$V = \{A \in \mathbb{R}^{3\times 3} : A^T = A\}$$

(a) (2 points) What properties should V satisfy to be a subspace?

Answer

(a) $\mathbf{0} \in V$
(b) If $A, B \in V$ then $A + B \in V$
(c) If $A \in V$ and $c \in \mathbb{R}$ then $cA \in V$.

(b) (3 points) Prove that V is a subspace.

Answer

(a) The zero vector in $\mathbb{R}^{3\times 3}$ is the 0-matrix. The transpose of that is again the zero matrix, so indeed, it is an element of V.
(b) We need to show that if $A^T = A$ and $B^T = B$ then $(A+B)^T = A + B$. Indeed,

$$(A+B)^T = A^T + B^T = A + B, \text{ assuming } A, B \in V.$$

(c) We need to show that if $A^T = A$ and $c \in \mathbb{R}$ then $(cA)^T = cA$. Indeed,

$$(cA)^T = c(A^T) = cA, \text{ assuming } A \in V.$$

(c) (2 points) What dimension does V have? Give a basis for V.

Answer

A general symmetric matrix is of the form

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$$

$$d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 6 given matrices are clearly linearly independent (each has a 1 in a position where all others have a 0).

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8. Let $\mathbf{v} \in \mathbb{R}^n$ and consider the map $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T(\mathbf{w}) = \mathbf{w} \cdot \mathbf{v}$.

(a) (2 points) What properties should T satisfy to be a linear transformation?

Answer

We should have that for all vectors $\mathbf{w}, \mathbf{u}$ in $\mathbb{R}^n$ that $T(\mathbf{w} + \mathbf{u}) = T(\mathbf{w}) + T(\mathbf{u})$. and that for any scalar $c \in \mathbb{R}$ we have that $T(c\mathbf{w}) = cT(\mathbf{w})$.

(b) (3 points) Prove that T is a linear transformation.

Answer

For any $\mathbf{w}, bu \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we have

$$T(\mathbf{w} + \mathbf{u}) = (\mathbf{w} + \mathbf{u}) \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} = T(\mathbf{w}) + T(\mathbf{u})$$

and

$$T(c\mathbf{w}) = (c\mathbf{w}) \cdot \mathbf{v} = c\,(\mathbf{w} \cdot \mathbf{v}) = cT(\mathbf{w})$$

which is what we have to prove.

(c) (3 points) Give the standard matrix of T and describe how you compute it.

Answer

The standard matrix of T is a matrix A such that $T(\mathbf{w}) = A\mathbf{w}$. Normally, one would compute this by taking the images of the standard vectors. However, in this situation, we know that

$$T(\mathbf{w}) = \mathbf{w} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

The last expression is indeed a matrix-vector product if we consider $\mathbf{v}^T$ as a matrix with one row and n columns. Hence $A = \mathbf{v}^T$ does the trick.

Bonus question: Let $W \subset \mathbb{R}^n$ be a subspace and let $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto W. Prove that proj_W can only have the eigenvalues 0 and 1 and describe the corresponding eigenspaces in terms of W.

Answer

In the lecture we have proved that applying an orthogonal projection twice does not acoomplish anything new:

$$\text{proj}_W(\text{proj}_W(\mathbf{y})) = \text{proj}_W(\mathbf{y})$$

If we have a non-zero vector $\mathbf{v}$ such that $\text{proj}_W(\mathbf{v}) = \lambda\mathbf{v}$ then

$$\lambda\mathbf{v} = \text{proj}_W(\mathbf{v}) = \text{proj}_W(\text{proj}_W(\mathbf{v})) = \text{proj}_W(\lambda\mathbf{v}) = \lambda\text{proj}_W(\mathbf{v}) = \lambda^2\mathbf{v}$$

This means that $\lambda = \lambda^2$, i.e., $\lambda = 0$ or $\lambda = 1$.

Remember that we can inquely write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^\perp$. With this notation, $\text{proj}_W(\mathbf{y}) = \hat{\mathbf{y}}$.

Hence, if $\mathbf{y}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$, then $\hat{\mathbf{y}} = \text{proj}_W(\mathbf{y}) = \mathbf{y}$, so $\mathbf{y} \in W$. Hence, the eigenspace corresponding to $\lambda = 1$ is W.

If $\mathbf{y}$ is an eigenvector for $\lambda = 0$ then $\hat{by} = \mathbf{0}$, so $\mathbf{z} = \mathbf{y} - \mathbf{0} = \mathbf{y}$. In that case, $\mathbf{y} \in W^\perp$, so $W^\perp$ is the eigenspace corresponding to $\lambda = 0$.