MACM 201 Fall 04

Midterm 2

Do not open this booklet until you are told to do so.

Last Name:

First Name:

Student Number:

Before the test begins, enter the information above.

Read all the questions before starting. You are **not allowed to write for the first two minutes** of the test.

Answer the questions in the order that best suits your strengths, taking into account the number of marks alloted to each question. Full marks will be reserved for answers that are **correct in all essential details**.

Remember that your answers should be in a form that another student could understand without undue effort: a poorly expressed but correct result is not sufficient.

This booklet should contain numbered pages 1 to 8, this page being page 1. As soon as the test begins, check to make sure you have all the pages and raise your hand if you do not.

1	2	3	4	5	6	7	Total
12	8	12	8	10	13	2	65

1. 15 children knock on my door on Hallowe'en asking for candies. I have a total of 60 identical candies to give to the children.

   (a) **3 marks** Define the *generalised binomial coefficient* $\binom{n}{i}$ for real n and integer $i \geq 0$.

   (b) **1 mark** State Newton's binomial theorem.

   (c) **5 marks** Using Newton's binomial theorem, determine the number of ways I can distribute the candies to the children so that no child goes away without a candy.

   (d) **3 marks** Suppose the children knock on my door in 3 groups of 5. How many ways can I distribute candies to the children so that each group of children receives a total of 20 candies and no child goes away without a candy?

2. (a) **4 marks** For $n \geq 1$, let a_n count the number of binary strings of length n that do not contain a run of 1's of odd length. For example, when $n = 5$ we include the binary string 01111 (which has a run of 1's of even length 4) but do not include the binary string 10110 (which has a run of 1's of even length 2 and another run of 1's of odd length 1). Find, with justification, a recurrence relation for a_n.

   (b) **4 marks** Solve the following recurrence relation by the standard method for homogeneous linear recurrence relations with constant coefficients:

$$2a_{n+2} = 11a_{n+1} - 5a_n, \quad \text{for } n \geq 0, \quad a_0 = 2,\ a_1 = -8.$$

3. Mark the following statements true or false, in either case giving a full and convincing justification for your answer.

   (a) **4 marks** The number of partitions of the positive integer n is the coefficient of x^n in $\prod_{i=1}^{\infty} \dfrac{1}{1-x^i}$.

   (b) **4 marks** The general solution to the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = n$ has the form $a_n = A + Bn + Cn^2$.

   (c) **4 marks** If there is a trail between two distinct vertices of an undirected graph then there is a path between them.

4. **8 marks** Up to isomorphism, how many loop-free undirected graphs are there with 9 vertices, where each vertex has degree 2? Justify your answer.

5. **10 marks** Solve the following recurrence relation **using the method of generating functions and making use of partial fractions**:

$$a_{n+2} - 3a_{n+1} + 2a_n = 0 \quad \text{for } n \geq 0,\ a_0 = 1,\ a_1 = 6.$$

6. Let G be an undirected graph or multigraph with no isolated vertices.

   (a) **2 marks** Assuming the definition of a trail is known, what is a *circuit* and what does it mean for G to have an *Euler circuit*?

   (b) **2 marks** State necessary and sufficient conditions for G to have an Euler circuit.

   (c) **4 marks** Prove that the conditions you have stated are necessary (in other words that if G has an Euler circuit then the conditions you have stated must hold).

   (d) **5 marks** Explain in careful detail why, if the conditions you have stated hold, G has a circuit of nonzero length containing any given vertex v.

7. **2 marks** Answer true or false: the elected winner in the 2004 U.S. presidential election is Al Gore.