SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Midterm 2 – Solutions

MACM 201 Summer 2007

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[2] **1.** (a) Evaluate $\binom{-\frac{3}{4}}{2}$.

$$\binom{-\frac{3}{4}}{2} = \frac{(-\frac{3}{4})(-\frac{3}{4}-1)}{2} = \frac{(-\frac{3}{4})(-\frac{7}{4})}{2} = \frac{21}{32}$$

[2] (b) Explain why $p(4) = 5$. (That is, list all objects that are counted by $p(4)$.)

$p(n)$ is the number of all partitions of n, so for $n = 4$ we have

- $4 = 4$
- $4 = 3 + 1$
- $4 = 2 + 2$
- $4 = 2 + 1 + 1$
- $4 = 1 + 1 + 1 + 1$

[4] (c) Let $a_n = 2$ and $b_n = n$ for all $n \geq 0$. Suppose that the sequence $c_0, c_1, c_2, \dots$ is the convolution of $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$. What is c_n?

$$\begin{aligned} c_n &= a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 \\ &= 2 \cdot n + 2 \cdot (n-1) + \cdots + 2 \cdot 0 \\ &= 2 \cdot \frac{n(n+1)}{2} = n(n+1). \end{aligned}$$

2. Find generating functions for the following sequences (include all the necessary computation).

[1] (a) $0, 1, 0, 1, 0, 1, 0, 1, \ldots$

$x^1 + x^3 + x^5 + x^7 + \cdots = \frac{x}{1-x^2}$ (geometric series with quotient x^2 and initial term x).
We might also use substitution.

[2] (b) $1, 0, 2, 0, 4, 0, 8, 0, 16, \ldots$

$1 + 2x^2 + 4x^4 + 8x^6 + 16x^8 + \cdots = \frac{1}{1-2x^2}$ (geometric series with quotient $2x^2$ and initial term 1)

[3] (c) $0, -1, 2, -3, 4, -5, 6, -7, \ldots$

The generating function for sequence $0, 1, 2, 3, \ldots$ is $\frac{x}{(1-x)^2}$. Using substitution $x := -x$ we get the generating function for the sequence under consideration: $\frac{-x}{(1+x)^2}$.

[2] (d) Find the coefficient of x^6 in $(1+2x)^{10}$.

By the binomial theorem $(1+2x)^{10} = \sum_{i=0}^{10} \binom{10}{i}(2x)^i$. Hence, $[x^6](1+2x)^{10} = \binom{10}{6}2^6$.

[4] (e) Find the coefficient of x^{54} in $x^5(x^3 + x^7 + x^{11} + x^{15} + \cdots)^3$.

$$\begin{aligned}
[x^{54}]x^5(x^3 + x^7 + x^{11} + x^{15} + \cdots)^3 &= [x^{54}]x^5 \cdot x^9(1 + x^4 + x^8 + x^{12} + x^{16} + \cdots)^3 \\
&= [x^{54}]x^{14}(1-x^4)^{-3} \\
&= [x^{40}](1-x^4)^{-3} \\
&= \binom{-3}{10}(-1)^{10} = \binom{3+10-1}{10} = \binom{12}{2}
\end{aligned}$$

[10] **3.** Let a_n be given by the following recurrence relation

$$a_n - 2a_{n-1} = 3^n \quad (n \geq 1), \qquad a_0 = 3\,.$$

Find the generating function for the sequence $a_0, a_1, a_2, \ldots$.

We multiply the equation for n by x^n and sum over all $n \geq 1$. This way we get

$$\sum_{n=1}^{\infty} a_n x^n - 2x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} 3^n x^n$$

After plugging in $f(x) = \sum_{n=0}^{\infty} a_n x^n$ we get

$$(f(x) - a_0) - 2xf(x) = \frac{3x}{1-3x}$$

We solve for $f(x)$, getting

$$f(x) = \frac{3}{1-2x} + \frac{3x}{(1-2x)(1-3x)}\,.$$

The following was not required, but for interest: Either by doing the partial fraction decomposition (which always works), or by putting both fraction over the same denominator, we get

$$f(x) = \frac{3}{1-3x}\,,$$

hence $a_n = 3^{n+1}$ (for all $n \geq 0$).

4. This question concerns finding the number of solutions to the equation

$$a + b + c = 100\,,$$

where a, b, c are nonnegative integers such that

- a is even,
- b is odd, and
- $c \geq 5$.

[3] (a) Express the answer as a coefficient of some power of x in an appropriate generating function $f(x)$.

The answer is $[x^{100}]f(x)$, where

$$\begin{aligned} f(x) &= (x^0 + x^2 + x^4 + \cdots)(x^1 + x^3 + x^5 + \cdots)(x^5 + x^6 + x^7 + \cdots) \\ &= \frac{1}{1-x^2}\frac{x}{1-x^2}\frac{x^5}{1-x} \end{aligned}$$

[4] (b) Express $f(x)$ as $\frac{\text{a polynomial}}{(1-x^2)^3}$.

By multiplying the last fraction by $\frac{1+x}{1+x}$ we get

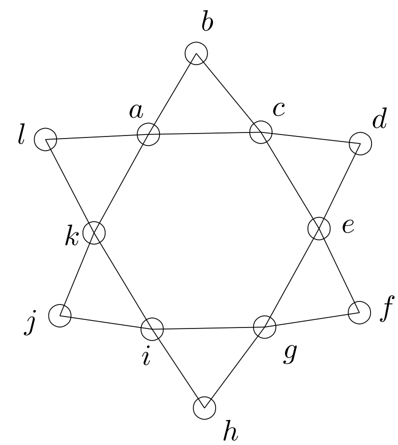
$$f(x) = \frac{x^6(1+x)}{(1-x^2)^3}$$

[3] (c) Use the binomial theorem to get the final answer.

As $\frac{1}{(1-x^2)^3}$ only contains even powers of x, we get

$$[x^{100}]f(x) = [x^{100}]\frac{x^6}{(1-x^2)^3} = [x^{94}]\frac{1}{(1-x^2)^3} = \binom{-3}{47}(-1)^{47} = \binom{49}{2}\,.$$

5. Consider various walks in the graph on the figure. In parts (b), (c), (e) explain why the walk you provided is not a path, a trail, and a cycle, respectively. (You may use any of the several ways to specify a walk.)



In each part, many solutions are possible!

[2] (a) Write down an a-f path.

$a - c - e - f$

[2] (b) Write down an a-f trail that is not a path.

$a - c - e - g - h - i - g - f$: this is not a path, as the vertex g is used twice. (No edge is repeated, though.)

[2] (c) Write down an a-f walk that is not a trail.

$a - c - d - e - c - d - e - f$: this is not a trail, as the edge $\{c, d\}$ is used twice.

[2] (d) Write down an a-a cycle.

$a - b - c - a$

[2] (e) Write down an a-a circuit that is not a cycle.

$a - b - c - a - k - l - a$: this is not a cycle, as the vertex a is repeated 'in the middle' (note that the 'repetition' at the end in part (d) does not matter).