SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Midterm 2 – Solutions

MACM 201 Spring 2007

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March 7, 2007, 12:30 – 13:20

[2] **1.** Solve the following recurrence relation

$$a_n = 10a_{n-1} \quad (n \geq 1), \qquad a_2 = 5\,.$$

We know that $a_n = A \cdot 10^n$ for some parameter A. By putting $n = 2$ we get $a_2 = 5 = A \cdot 10^2$, hence $A = 1/20$, and

$$a_n = \frac{1}{20} 10^n \qquad (n \geq 0)\,.$$

[5] **2.** (a) Solve the following recurrence relation. (More space is available on the next page.)

$$a_{n+2} + 4a_{n+1} + 4a_n = 0 \quad (n \geq 0), \qquad a_0 = 1, a_1 = 4\,.$$

The characteristic equation is $t^2 + 4t + 4 = 0$ and as $t^2 + 4t + 4 = (t + 2)^2$, we have repeated root $t = -2$. So, the general solution is of form $a_n = A(-2)^n + Bn(-2)^n$. We plug in $n = 0, 1$:

$$\begin{aligned} 1 &= a_0 = A \cdot 1 + B \cdot 0, \qquad \text{hence } A = 1 \\ 4 &= a_1 = A \cdot (-2) + B \cdot (-2), \qquad \text{hence } B = -3 \end{aligned}$$

It follows that

$$a_n = (-2)^n - 3n(-2)^n \qquad (n \geq 0)\,.$$

[2] (b) Check the answer for $n = 2$.

Using the recurrence, we have $a_2 = -4a_1 - 4a_0 = -4 \cdot 4 - 4 \cdot 1 = -20$.

Using the formula we derived, we see $a_2 = (-2)^2 - 3 \cdot 2(-2)^2 = 4 - 24 = -20$.

[10] **3.** For $n \geq 0$ let a_n be the number of ways n can be expressed as a sum of **odd** integers, if order matters. For example, $a_4 = 3$, as $4 = 3 + 1 = 1 + 3 = 1 + 1 + 1 + 1$.

Find a recurrence relation for a_n (do not solve it).

We group all expressions for n according to the last term.

1. $n = (\dots) + 1$ (the last term is 1). These expressions correspond precisely to expressions $n - 1 = (\dots)$, that is to different ways to write $n - 1$ as a sum of odd integers. There are a_{n-1} ways to do this.

2. $n = (\dots) + k$ where $k > 1$ (the last term is not 1). These expressions precisely correspond to expressions $n - 2 = (\dots) + (k - 2)$, that is to different ways to write $n - 2$ as a sum of odd integers. There are a_{n-2} ways to do this.

So, we have

$$a_n = a_{n-1} + a_{n-2}, \qquad (n \geq 2)\,.$$

We still need the initial conditions: $1 = 1$ is the only expression for 1, $2 = 1+1$ the only expression for 2. Thus,

$$a_1 = a_2 = 1\,.$$

For $n = 0$ it is not easy to make one's mind if there is 0 or 1 way, but the recurrence relation gives $a_0 = 0$.

From this we can come with a short solution (this was **not** asked for in the exam): a_n satisfies the same recurrence relation and the same initial conditions as the Fibonacci numbers, so $a_n = F_n$. Consequently,

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right) \qquad (n \geq 0)\,.$$

4. Find $\sum_{k=1}^{n} k2^k$ by means of recurrence relations. (Use the back of the previous page if more space is needed.)

[2] (a) Put $a_n = \sum_{k=1}^{n} k2^k$ and write a recurrence relation for a_n.

$a_n = a_{n-1} + n2^n$ for $n \geq 1$.

[2] (b) Find a_1, a_2, and a_3 'by hand'.

$a_1 = 1 \cdot 2^1 = 2$
$a_2 = 1 \cdot 2^1 + 2 \cdot 2^2 = 2 + 8 = 10$
$a_2 = 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 = 2 + 8 + 24 = 34$

[5] (c) Solve the recurrence relation found in (a).

1. The homogeneous solution is $a_n^{(h)} = A$ (A is any constant).
2. We will look for a particular solution in form $a_n^{(p)} = (Bn + C)2^n$ (B, C suitable constants). Plugging in the recurrence relation we obtain

$$2^n(Bn + C) = 2^{n-1}(Bn + C - B) + n2^n\,.$$

A simple (in fact, the only) way how to satify this equation for every n is solving a system of two equations

$$\begin{aligned} n2^nB &= 2^n n(B/2 + 1) \\ 2^nC &= 2^n(C - B)/2 \end{aligned}$$

This gives us $B = 2$, $C = -B = -2$. We've found $a_n^{(p)} = 2(n-1)2^n$.

3. The general solution is $a_n = a_n^{(h)} + a_n^{(p)} = A + 2(n-1)2^n$. To determine A we let $n = 1$: $2 = a_1 = A + 0$, hence $A = 2$.

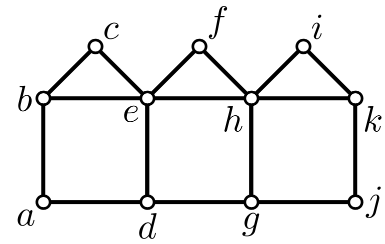
Altogether we have

$$a_n = (n-1)2^{n+1} + 2 \qquad (n \geq 0)\,.$$

[1] (d) Verify the solution for $n = 1, 2, 3$.

$a_1 = 0 \cdot 2^2 + 2 = 2$
$a_2 = 1 \cdot 2^3 + 2 = 10$
$a_3 = 2 \cdot 2^4 + 2 = 34$

5. Consider various walks in the graph on the figure. In parts (b), (c), (e) explain why the walk you provided is not a path, a trail, and a cycle, respectively.



[2] (a) Write down an a-f path.

Using various notations, we can write one such path as:

- $a - d - e - f$
- $a, \{a, d\}, d, \{d, e\}, e, \{e, f\}, f$
- a, ad, d, de, e, ef, f
- $\{a, d\}, \{d, e\}, \{e, f\}$
- ad, de, ef

In the next parts, we will only use the first notation.

[2] (b) Write down an a-f trail that is not a path.

$a - b - e - h - g - d - e - f$ (this is not a path, as the vertex e is used twice).

[2] (c) Write down an a-f walk that is not a trail.

$a - d - g - h - e - d - g - h - f$ (this is not a trail, as the edge $\{d, g\}$ is used twice).

[2] (d) Write down an a-a cycle.

$a - b - e - d - a$

[2] (e) Write down an a-a circuit (closed trail) that is not a cycle.

$a - b - e - h - f - e - d - a$ (this is not a cycle, as the vertex e is used twice).