Simon Fraser University
Department of Mathematics
Burnaby Campus

**MACM 201-**3, Fall 2005
Midterm 1
October 5th, 2005, 12:30 pm – 1:20 pm

Last Name (please print): ______________________________

First Name (please print): ______________________________

Student Number: ______________________________

Signature: ______________________________

## Instructions:

1. Do NOT open this booklet yet.
2. Fill in the above box NOW.
3. This exam contains 6 pages with a total of 5 questions. Once the exam begins please check your exam is complete.
4. No notes are allowed. Cellphones off!
5. If you run out of space in a problem, use the space on the back of the previous page and indicate where the solution continues.
6. For the first 2 minutes of the test you are NOT ALLOWED to write, except to mark key words of questions. Use this time to read the questions carefully and decide the order in which you will answer them.
7. Full marks will be reserved for answers that are CORRECT IN ALL ESSENTIAL DETAILS, and could be understood by another student without undue effort.
8. During the examination, speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is FORBIDDEN.
9. You MUST stop writing when time is called.

**Do not write in this table!**

| **Question** | **Marks** |
|---|---|
| 1 | **/10** |
| 2 | **/8** |
| 3 | **/8** |
| 4 | **/8** |
| 5 | **/6** |
| **Total** | **/40** |

1. Let $S$ be a set and let $c_1, c_2, \ldots c_t$ be conditions on the elements of $S$. A student writes the Principle of Inclusion-Exclusion for $S$ incorrectly as:

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\ldots\overline{c_t}) = N + \sum_{i} N(c_i) + \sum_{i,j} N(c_i c_j) + \sum_{i,j,k} N(c_i c_j c_k) + \ldots + N(c_1 c_2 c_3 \ldots c_t).$$

   (a) **[2 marks]** Mark all necessary changes to the above formula so that it is a correct statement of the Principle of Inclusion-Exclusion.

   (b) **[1 mark]** What is meant by the expression $N(c_i\, c_j)$?

   (c) **[2 marks]** Let $x$ be an element of $S$ that satisfies exactly $r$ of the conditions $c_1, c_2, \ldots c_t$. How many times is $x$ counted in the expression $\sum_{i,j} N(c_i c_j)$? Explain your answer.

   (d) **[5 marks]** Five students take a midterm test in a room containing 8 chairs. When the test is over they go for coffee, then return to the same room to compare answers. In how many ways can the students seat themselves on returning from coffee so that no student sits in the same chair he or she occupied during the test? (You should give a numerical expression for your answer but you need not evaluate it as a single number.)

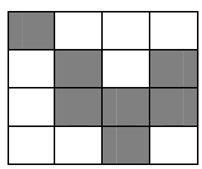
2. Let $r_k(C)$ be the number of ways of placing $k \geq 0$ rooks on the shaded squares of a chessboard $C$ such that no two rooks lie in the same row or column. A recursive formula for $r_k(C)$ is:

$$r_k(C) = r_{k-1}(C_A) + r_k(C_s).$$

(a) **[2 marks]** What is meant by the expressions $C_A$ and $C_s$ in this formula?

(b) **[2 marks]** Explain why this recursive formula holds.

(c) **[4 marks]** Calculate the rook polynomial for the shaded chessboard shown. You may express your answer as a function of *x* without necessarily calculating each coefficient of the rook polynomial explicitly.



3.

(a) **[2 marks]** Define the *generalised binomial coefficient* $\binom{n}{i}$ for real $n$ and integer $i \geq 0$.

(b) **[1 mark]** State Newton's binomial theorem.

(c) **[5 marks]** Find the coefficient of $x^{83}$ in $(x^5 + x^8 + x^{11} + x^{14} + x^{17})^{10}$. You may express your answer in terms of (regular) binomial coefficients without necessarily evaluating it as a single number.

4.

(a) **[3 marks]** Find and simplify the generating function for the number of partitions of a positive integer *n* where no summand appears more than twice.

(b) **[3 marks]** Find and simplify the generating function for the number of partitions of a positive integer *n* where no summand is divisible by 3.

(c) **[2 marks]** Use your answers to parts (a) and (b) to conclude that the number of partitions of a positive integer *n* where no summand appears more than twice is equal to the number of partitions of *n* where no summand is divisible by 3.

5. The following statements are both FALSE. In both cases demonstrate conclusively why.

(a) **[3 marks]** Let $f(x)=\sum_{i=0}^{\infty}a_i x^i$, let $g(x)=\sum_{i=0}^{\infty}b_i x^i$, and let $h(x)=f(x)g(x)$. Then we can write $h(x)=\sum_{i=0}^{\infty}c_i x^i$, where $c_i=a_i b_i$ for all *i*.

(b) **[3 marks]** There are no more than 10 derangements of 1, 2, 3, 4, 5 where 1, 2, 3 appear in the first four positions.