# MACM 201 Fall 04

## Midterm 1

**Do not open this booklet until you are told to do so**.

Last Name:

First Name:

Student Number:

Before the test begins, enter the information above.

Read all the questions before starting. You are **not allowed to write for the first two minutes** of the test.

Answer the questions in the order that best suits your strengths, taking into account the number of marks alloted to each question. Full marks will be reserved for answers that are **correct in all essential details**.

Remember that your answers should be in a form that another student could understand without undue effort: a poorly expressed but correct result is not sufficient.

This booklet should contain numbered pages 1 to 7, this page being page 1. As soon as the test begins, check to make sure you have all the pages and raise your hand if you do not.

| 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|
| | | | | | | |
| 6 | 5 | 6 | 11 | 10 | 2 | 40 |

1. (a) **1 mark** Let $A$ and $B$ be events. Define the *conditional probability* $P(A|B)$.

   (b) **5 marks** Each of five friends in turn tosses a coin whose probability of landing heads is $p$. If we know that not all five tosses land the same, what is the probability that one toss lands differently from all the others?

2. **5 marks** Monty has devised a new game, involving one lockable door behind which is a car. The door is initially unlocked with probability $\frac{1}{3}$. Monty offers the contestant a box of 12 keys, only one of which fits the lock of the door. The contestant is allowed to choose two keys from the box. What is the probability the contestant can open the door to claim the car?

3. Each of the following statements is false. For statements (a), (b) and (c), *correct* the text contained within the box. For statement (d), *demonstrate conclusively* that it is false.

   (a) **1 mark** The number of derangements of $n$ objects is $\boxed{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots}$

   (b) **1 mark** The number of integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$, where $x_i \geq 0$ for each $i$, is $\boxed{\binom{n+r-1}{r-1}}$.

   (c) **1 mark** The Generalised Principle of Inclusion-Exclusion gives an expression for $\boxed{\text{the number of elements of the set satisfying at least } m \text{ conditions}}$.

   (d) **3 marks** Let $S$ be the set of shaded chessboards comprising 4 squares. Let $T$ be the set of rook polynomials corresponding to the chessboards of $S$. There is only one polynomial of degree 3 in $T$.

4. (a) **2 marks** Define the *rook polynomial* $r(C, x)$ for a chessboard $C$ consisting of a set of shaded squares, defining any notation you use.

   (b) **2 marks** State a recursive formula for $r(C, x)$, defining any notation you use.

   (c) We wish to determine the number of $1-1$ functions $f : \{1, 2, 3\} \to \{u, v, w, x\}$ which satisfy the constraints

$$f(1) \neq u, x \quad f(2) \neq v \quad f(3) \neq u, v, w.$$

   **2 marks** Set up this problem as an inclusion-exclusion problem by defining a suitable set $S$ and conditions $c_i$, and represent the conditions by means of some chessboard $C$.

   **3 marks** Use the recursive formula to evaluate the rook polynomial $r(C, x)$ for the chessboard $C$ you have drawn.

   **2 marks** Use the Principle of Inclusion-Exclusion applied to $1-1$ functions (which you need not prove or justify), and the calculated value of $r(C, x)$, to determine the number of $1-1$ functions satisfying the given constraints.

5. Let $S$ be a set of $N$ elements and let $c_1, c_2, \ldots, c_t$ be conditions on elements of $S$.

   (a) **2 marks** State the *Principle of Inclusion-Exclusion*, defining any notation you use.

   (b) **8 marks** Prove the Principle of Inclusion-Exclusion.

6. Note: this question will be marked in accordance with your preference stated below.

   **2 marks** Answer yes or no: would you like to be awarded 2 marks for answering this question?