SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Midterm 1 – Solutions

MACM 201 Summer 2007

Instructor: Robert Šámal

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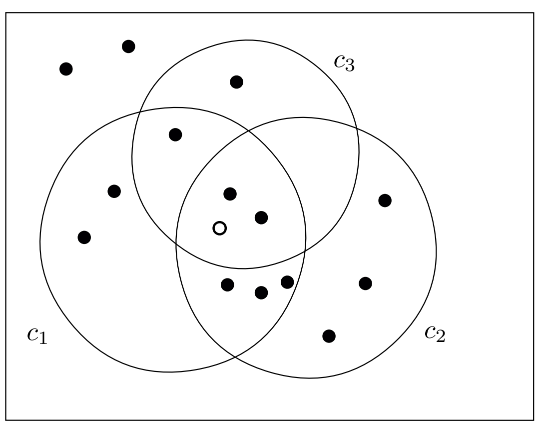
1. Let S be a (finite) set, $c_1, \ldots, c_t$ conditions on the elements of S.

[1] (a) Suppose $t = 3$. State the Principle of Inclusion and Exclusion, and explain the used notation. Don't use '$\ldots$', nor '$\sum$'.

$$\begin{aligned} N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ ) &= |S| - \big(N(c_1) + N(c_2) + N(c_3)\big) \\ &\quad + \big(N(c_1c_2) + N(c_1c_3) + N(c_2c_3)\big) \\ &\quad - N(c_1c_2c_3) \end{aligned}$$

Here (as usually), $N(c_i)$ is the number of elements of S satisfying c_i, $N(c_ic_j)$ the number of elements satisfying c_i and c_j, and $N(c_1c_2c_3)$ the number of elements satisfying all three conditions. Finally, $N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ )$ is the number of elements satisfying neither of them.

[3] (b) Let $t = 3$ again, and let S be as on the figure (both full and empty points count). Fill in the following table:



$N(c_1)$	$N(c_2)$	$N(c_3)$	$N(c_1c_2)$	$N(\ \overline{c_2}\ \ c_3\ )$	S_1
9	9	5	6	2	23

[1] (c) With the same set S as in the previous part, how many times is the element of S denoted by empty circle counted in the sum $S_2 = \sum_{1 \le i < j \le t} N(c_ic_j)$?

3 times

Decide if the following formulas are true or false, and if false, correct them!

[1] (d) $N(\ \overline{c_1}\ ) = N - N(\ c_1\ )$

True.

[1] (e) $N(\ \overline{c_1}\ \ c_2\ ) = N(\ c_2\ ) - N(\ c_1\ \ c_2\ )$

True.

[1] (f) $N(\ c_1\ \ c_2\ ) = N(\ c_2\ ) - N(\ c_1\ \ \overline{c_2}\ )$

False. Possible corrections:

$N(\ c_1\ \ c_2\ ) = N(\ c_1\ ) - N(\ c_1\ \ \overline{c_2}\ )$

$N(\ c_1\ \ c_2\ ) = N(\ c_2\ ) - N(\ \overline{c_1}\ \ c_2\ )$

[2] (g) $N(\ c_1\ \ c_2\ ) = N - N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ ) - N(\ \overline{c_1}\ \ \overline{c_2}\ \ c_3\ )$

False. Possible corrections include:

$N(\ c_1$ or $c_2\ ) = N - N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ ) - N(\ \overline{c_1}\ \ \overline{c_2}\ \ c_3\ )$

$N(\ c_1\ \ c_2\ ) = N - (N(\ \overline{c_1}\ ) + N(\ \overline{c_2}\ )) + (N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ ) + N(\ \overline{c_1}\ \ \overline{c_2}\ \ c_3\ ))$

$N(\ c_1\ \ c_2\ ) = N - N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ ) - N(\ \overline{c_1}\ \ \overline{c_2}\ \ c_3\ ) - N(\ c_1\ \ \overline{c_2}\ ) - N(\ \overline{c_1}\ \ c_2\ )$

2. Use Principle of Inclusion and Exclusion to determine, how many integers n are there so that $1 \leq n \leq 211$ and

[5] (a) n is not divisible by any of 2, 5, 7?

We will have $S = \{1, 2, \ldots, 211\}$, and conditions c_1: is divisible by 2, c_2: is divisible by 5, and c_3: is divisible by 7. We will repeatedly use the following fact: The number of integers among 1, 2, ..., 211 that are divisible by an integer k is $\lfloor 211/k \rfloor$. Also note, that an integer is divisible by 2 and 5 iff it is divisible by 10.

- $S_0 = |S| = 211$
- $S_1 = N(c_1) + N(c_2) + N(c_3) = \lfloor 211/2 \rfloor + \lfloor 211/5 \rfloor + \lfloor 211/7 \rfloor = 105 + 42 + 30 = 177$
- $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = \lfloor 211/10 \rfloor + \lfloor 211/14 \rfloor + \lfloor 211/35 \rfloor = 21 + 15 + 6 = 42$
- $S_3 = N(c_1c_2c_3) = \lfloor 211/70 \rfloor = 3$

Now, the requested number is

$$N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ ) = S_0 - S_1 + S_2 - S_3 = 211 - 177 + 42 - 3 = 73\,.$$

[5] (b) n is divisible by at least two of 2, 5, 7?

We will use the same set S and conditions c_1, c_2, c_3 as in the first part. Then we use the generalization of the Principle of Inclusion and Exclusion:

$$L_2 = S_2 - 2S_3 = 42 - 2 \cdot 3 = 36\,.$$

[10] **3.** Two giants, three witches, and seven dwarfs are going to take a group photograph, that is they need to arrange in a single row. To make the photograph look interesting, they want that

- the two giants don't stand next to each other,
- the three witches don't stand all together, **and**
- the seven dwarfs don't stand all together.

For example, the following is a proper arrangement: $G_2D_7D_2D_3W_3D_5D_6D_4D_1W_2W_1G_1$.
How many different photographs can they make in this way?

The set S will consist of all possible arrangements of the creatures, that is of all permutations of $2+3+7=12$ objects. (We consider the creatures of the same 'type' to be distinguishable. If we suppose they are not, we get another problem, but the solution is similar.) The conditions will be

- c_1: the two giants **do** stand next to each other,
- c_2: the three witches **do** stand all together,
- c_3: the seven dwarfs **do** stand all together.

Let's find the numbers now. As a typical case, consider $N(c_2c_3)$ first. We know that all witches and all dwarfs stand together: that is we are looking at permutations of 4 'objects' (two dwarfs, a group of witches and a group of dwarfs), and we also have to consider the permutations of the dwarfs and of the witches. So, we get $N(c_2c_3) = 4! \cdot 3! \cdot 7!$. Let's go systematic now.

1. $S_0 = 12!$
2. $S_1 = N(c_1) + N(c_2) + N(c_3)$, where
   - $N(c_1) = 11! \cdot 2!$ (we make the two giants into one group, which makes for 11 'objects' to permute).
   - $N(c_2) = 10! \cdot 3!$ (we make the three witches into one group)
   - $N(c_3) = 6! \cdot 7!$ (we make the dwarfs into one group)
3. $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3)$, where
   - $N(c_1c_2) = 9! \cdot 2! \cdot 3!$
   - $N(c_1c_3) = 5! \cdot 2! \cdot 7!$
   - $N(c_2c_3) = 4! \cdot 3! \cdot 7!$
4. $S_3 = N(c_1c_2c_3) = 3! \cdot 2! \cdot 3! \cdot 7!$

We are ready for the final answer. The number of proper arrangements is

$$N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) = S_0 - S_1 + S_2 - S_3 = 12! - (11!\cdot 2! + 10!\cdot 3! + 6!\cdot 7!) + (9!\cdot 2!\cdot 3! + 5!\cdot 2!\cdot 7! + 4!\cdot 3!\cdot 7!) - 3!\cdot 2!\cdot 3!\cdot 7!$$

To get the idea, this equals $379\,693\,440$ (while $12! = 479\,001\,600$), but you were not supposed to enumerate it.

[3] **4.** (a) Solve the following recurrence relation

$$a_n = 4a_{n-1} \quad (n \geq 1), \qquad a_2 = 8\,.$$

We know that $a_n = A \cdot 4^n$ for some parameter A. By putting $n = 2$ we get $a_2 = 8 = A \cdot 4^2$, hence $A = 1/2$, and

$$a_n = \frac{1}{2}4^n \qquad (n \geq 0)\,.$$

[5] (b) Solve the following recurrence relation. (More space is available on the next page.)

$$a_{n+2} - 6a_{n+1} + 8a_n = 0 \quad (n \geq 0), \qquad a_0 = 1, a_1 = 3\,.$$

The characteristic equation is $t^2 - 6t + 8 = 0$ and as $t^2 - 6t + 8 = (t-2)(t-4)$, we have roots 2 and 4. So, the general solution is of form $a_n = A2^n + B4^n$. We plug in $n = 0$, 1:

$$1 \quad = a_0 = A \cdot 1 + B \cdot 1, 3 \quad = a_1 = A \cdot 2 + B \cdot 4.$$

By solving this system of equations we get $A = B = 1/2$. It follows that

$$a_n = \frac{1}{2}2^n + \frac{1}{2}4^n \qquad (n \geq 0)\,.$$

[2] (c) Check the answer for $n = 2$.

Using the recurrence, we have $a_2 = 6a_1 - 8a_0 = 6 \cdot 3 - 8 \cdot 1 = 10$.

Using the formula we derived, we get $a_2 = (2^2 + 4^2)/2 = 10$.

5. Find $\sum_{k=1}^{n} \frac{k}{2^k}$ be means of recurrence relations. (Use the back of the previous page if more space is needed.)

[2] (a) Put $a_n = \sum_{k=1}^{n} \frac{k}{2^k}$ and write a recurrence relation for a_n.

$a_n = a_{n-1} + \frac{n}{2^n}$ for $n \geq 1$.

[2] (b) Find a_1, a_2, and a_3 'by hand'.

$a_1 = 1/2^1 = 1/2$
$a_2 = 1/2^1 + 2/2^2 = 1$
$a_2 = 1/2^1 + 2/2^2 + 3/2^3 = 11/8$

[5] (c) Solve the recurrence relation found in (a).

1. The homogeneous solution is $a_n^{(h)} = A$ (A is any constant).
2. We will look for a particular solution in form $a_n^{(p)} = (Bn + C)/2^n$ (B, C suitable constants). Plugging in the recurrence relation we obtain

$$\frac{Bn + C}{2^n} - \frac{B(n-1) + C}{2^{n-1}} = \frac{n}{2^n}\,.$$

and after some algebra we get

$$-B \cdot n + (2B - C) = n$$

A simple (in fact, the only) way how to satify this equation for every n is solving a system of two equations

$$\begin{aligned} -B \cdot n &= n \\ 2B - C &= 0 \end{aligned}$$

This gives us $B = -1$, $C = -2$. We've found $a_n^{(p)} = -(n+2)/2^n$.

3. The general solution is $a_n = a_n^{(h)} + a_n^{(p)} = A - (n+2)/2^n$. To determine A we let $n = 1$: $1/2 = a_1 = A - 3/2$, hence $A = 2$.

Altogether we have

$$a_n = 2 - \frac{n+2}{2^n} \qquad (n \geq 0)\,.$$

[1] (d) Verify the solution for $n = 1, 2, 3$.

$a_1 = 2 - 3/2^1 = 1/2$
$a_2 = 2 - 4/2^2 = 1$
$a_3 = 2 - 5/2^3 = 11/8$,
so it works.