SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Midterm 1 – Solutions

MACM 201 Spring 2007

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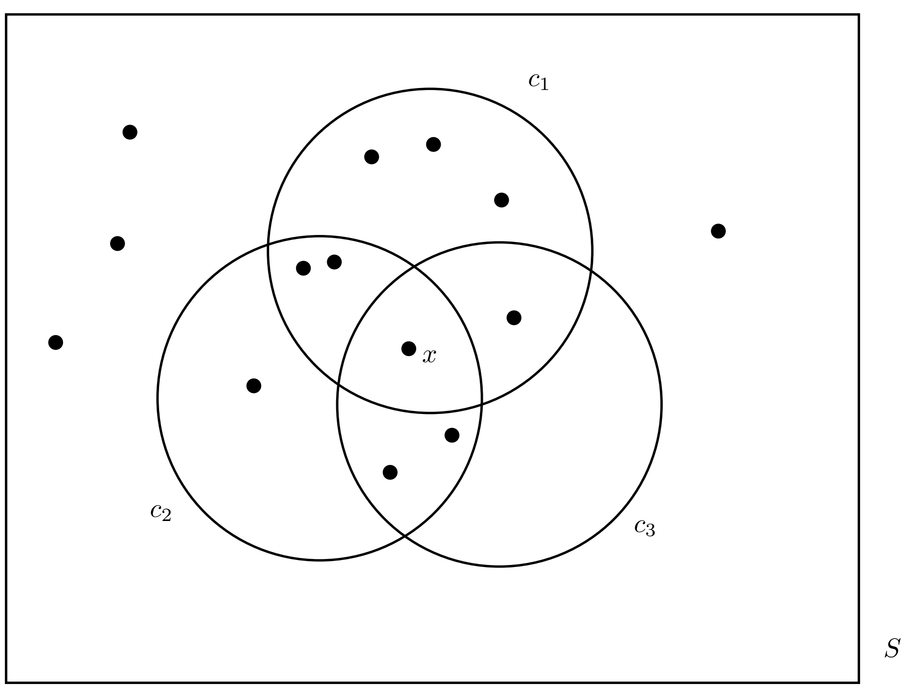
February 7, 2007, 12:30 – 13:20

1. Let S be a (finite) set, $c_1, \dots, c_t$ conditions on the elements of S.

[1] (a) Suppose $t = 4$. State the Principle of Inclusion and Exclusion, and explain the used notation. Don't use '. . . ', nor '$\sum$'.

$$\begin{aligned} N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}) = {} & |S| - \big(N(c_1) + N(c_2) + N(c_3) + N(c_4)\big) \\ & + \big(N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)\big) \\ & - \big(N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)\big) \\ & + N(c_1c_2c_3c_4) \end{aligned}$$

[3] (b) Let $t = 3$ and let S be as on the figure, fill in the following table:



$N(c_1)$	$N(c_2)$	$N(c_3)$	$N(c_1c_2)$	$N(c_1\overline{c_2})$	S_2
7	6	4	3	4	8

For the last term: $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = 3 + 2 + 3 = 8.$

[1] (c) With the same set S as in previous part, how many times is the element x counted in the sum $S_2 = \sum_{1\le i<j\le t} N(c_ic_j)$?

Three times: in each of $N(c_1c_2)$, $N(c_1c_3)$, and $N(c_2c_3)$.

[5] (d) Suppose $t = 6$. Write and justify a formula for $N(c_1\ \overline{c_2}\ \overline{c_3}\ c_4c_5\ \overline{c_6})$! In the formula you may use only the numbers that appear in the Principle of Inclusion and Exclusion, e.g., $N(c_1)$, $N(c_1c_2)$, but not $N(c_1\overline{c_2})$.

$$
\begin{aligned}
N(\ c_1\ \ \overline{c_2}\ \ \overline{c_3}\ \ c_4\ \ c_5\ \ \overline{c_6}\ ) \ \ = \ \ & N(c_1c_4c_5) \\
& -\big(N(c_1c_2c_4c_5) + N(c_1c_3c_4c_5) + N(c_1c_4c_5c_6)\big) \\
& +\big(N(c_1c_2c_3c_4c_5) + N(c_1c_2c_4c_5c_6) + N(c_1c_3c_4c_5c_6)\big) \\
& -N(c_1c_2c_3c_4c_5c_6)
\end{aligned}
$$

To justify the formula, put $S' = \{x \in S; x$ satisfies $c_1c_4c_5\}$ and define for elements of S' conditions $c_1' = c_2$, $c_2' = c_3$, and $c_3' = c_6$. It remains to use PIE for the set S' and conditions c_1', c_2', and c_3'. We only have to observe, that

- $|S'| = N(c_1c_4c_5)$
- $N(c_1') = N(c_1c_2c_4c_5)$
- $N(c_2') = N(c_1c_3c_4c_5)$
- $N(c_1'c_2') = N(c_1c_2c_3c_4c_5)$
- $\ldots$
- $N(\ \overline{c_1'}\ \ \overline{c_2'}\ \ \overline{c_3'}\ ) = N(\ c_1\ \ \overline{c_2}\ \ \overline{c_3}\ \ c_4\ \ c_5\ \ \overline{c_6}\ )$

2. Use Principle of Inclusion and Exclusion to determine, how many integers n are there (you should write down an expression but don't have to evaluate it) so that $1 \leq n \leq 2007$ and

[5] (a) n is not a second, third, or fifth power?

We will have $S = \{1, 2, \ldots, 2007\}$, and conditions c_1: n is a second power ($n = k^2$ for some integer k) c_2: n is a third power ($n = k^3$ for some integer k) c_3: n is a fifth power ($n = k^5$ for some integer k) We will repeatedly use the following fact: The number of integers among 1, 2, ..., 2007 that are an t-th power is $\lfloor \sqrt[t]{2007} \rfloor$. Also note, that an integer is both a 2nd and a 3rd power iff it is a 6th power; it is both a 2nd and a 5th power iff it is a 10th power, etc.

- $S_0 = |S| = 2007$
- $S_1 = N(c_1) + N(c_2) + N(c_3) = \lfloor \sqrt{2007} \rfloor + \lfloor \sqrt[3]{2007} \rfloor + \lfloor \sqrt[5]{2007} \rfloor = 44 + 12 + 4 = 60$
- $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = \lfloor \sqrt[6]{2007} \rfloor + \lfloor \sqrt[10]{2007} \rfloor + \lfloor \sqrt[15]{2007} \rfloor = 3 + 2 + 1 = 6$
- $S_3 = N(c_1c_2c_3) = \lfloor \sqrt[30]{2007} \rfloor = 1$

Now, the requested number is

$$N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ ) = S_0 - S_1 + S_2 - S_3 = 2007 - 60 + 6 - 1 = 1952\,.$$

(The actual numbers were not required, a formula with the $\sqrt{}$'s is fine.)

[5] (b) n is not a second, third, or fourth power?

If n is a fourth power, then it is a second power as well. Hence, we can use PIE for only two of the conditions of part (a): for c_1 and c_2.

- $S_0 = |S| = 2007$
- $S_1 = N(c_1) + N(c_2)) = \lfloor \sqrt{2007} \rfloor + \lfloor \sqrt[3]{2007} \rfloor = 44 + 12 = 56$
- $S_2 = N(c_1c_2) = \lfloor \sqrt[6]{2007} \rfloor = 3$

Now, we obtain the answer:

$$N(\ \overline{c_1}\ \ \overline{c_2}\ ) = S_0 - S_1 + S_2 = 2007 - 56 + 3 = 1954\,.$$

(The actual numbers were not required, a formula with the $\sqrt{}$'s is fine.)

3. How many ways are there to distribute n undistinguishable objects among k people if

[2] (a) everybody can receive any amount of them?

This is the same as solving the equation

$$a_1 + a_2 + \cdots + a_k = n$$

with integers $a_i \geq 0$. We mentioned in class (a result from MACM 101) that this equals $\binom{n+k-1}{k-1}$.

Another way how to get this result is to find that the answer is $[x^n](1-x)^{-k}$ and use generalized binomial theorem.

[4] (b) everybody should receive at least one of them?

Now we replace the condition $a_i \geq 0$ by $a_i \geq 1$. Put $b_i = a_i - 1$ and observe that

$$b_1 + b_2 + \cdots + b_k = n - k,$$

and $b_i \geq 0$ are integers. Hence, the number of solutions is $\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$.

Again, a possible alternative is to express $[x^n]\left(\frac{x}{1-x}\right)^k$ by the use of generalized binomial theorem.

[10] (c) everybody shoud receive at most $n/3$ of them? (Suppose n is divisible by 3.)

We will use PIE. Let $S = \{(a_1, \ldots, a_k) \in \mathbb{N}^k, a_1 + \cdots + a_k = n\}$. For each $i = 1, \ldots, k$ we let c_i be the condition on elements of S saying $a_i > n/3$, equivalently $a_i \geq n/3 + 1$.

To find $N(c_i)$ we let $b_j = a_j$ for $j \neq i$ and $b_j = a_j - n/3 - 1$ for $j = i$. Every solution $(a_1, \ldots, a_k)$ satisfying c_i transforms to a solution to

$$b_1 + b_2 + \cdots + b_k = n - n/3 - 1,$$

with nonnegative integers b_j. Hence, $N(c_i) = \binom{2n/3-1+k-1}{k-1}$ and $S_1 = \sum_{i=1}^{k} N(c_i) = k\binom{2n/3-1+k-1}{k-1}$.

With a pair of conditions c_i, c_l we put $b_j = a_j$ if $j \neq i$, $j \neq l$ and $b_j = a_j - n/3 - 1$ otherwise. Again, we transform solutions satisfying $c_i c_l$ to a solution to

$$b_1 + b_2 + \cdots + b_k = n - 2(n/3 + 1),$$

with nonnegative integers b_j. Hence, $N(c_i c_j) = \binom{n/3-2+k-1}{k-1}$ and $S_2 = \sum_{i=1}^{k} \sum_{l=i+1}^{k} N(c_i c_j) = \binom{k}{2}\binom{n/3-2+k-1}{k-1}$.

It is not possible for three variables to be greater than $n/3$ and still sum to n, hence $S_3 = S_4 = \cdots = S_k = 0$.

Therefore, the required number is

$$S_0 - S_1 + S_2 = \binom{n+k-1}{k-1} - k\binom{2n/3-1+k-1}{k-1} + \binom{k}{2}\binom{n/3-2+k-1}{k-1}.$$

4. Find generating functions for the following sequences (include all the necessary computation).

[1] (a) $0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$

$x^3 + x^5 + x^7 + x^9 + x^{11} + \cdots = \frac{x^3}{1-x^2}$ (geometric series with quotient x^2)

[2] (b) $0, 0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2, \ldots$

$2x^2 - 2x^4 + 2x^6 - 2x^8 + 2x^{10} - 2x^{12} + \cdots = \frac{2x^2}{1+x^2}$ (geometric series with quotient $-x^2$)

[3] (c) $4, 5, 6, 7, 8, 0, 16, 0, 32, 0, 64, 0, 128, 0, 256, 0, 512, 0, \ldots$

$4 + 5x + 6x^2 + 7x^3 + 8x^4 + 16x^6 + 32x^8 + 64x^{10} + 128x^{12} + 256x^{14} + 512x^{16} + \cdots = 4 + 5x + 6x^2 + 7x^3 + \frac{8x^4}{1-2x^2}$

(we have a polynomial plus a geometric series with quotient $2x^2$)

[3] (d) $0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, \ldots$

$$\begin{aligned} & x + 2x^2 + x^4 + 2x^5 + x^7 + 2x^8 + x^{10} + 2x^{11} + \cdots \\ = \; & x + x^4 + x^7 + x^{10} + \cdots + 2x^2 + 2x^5 + 2x^8 + 2x^{11} + \cdots \\ = \; & \frac{x}{1-x^3} + \frac{2x^2}{1-x^3} = \frac{x+2x^2}{1-x^3}\,. \end{aligned}$$

(we see a sum of two geometric series with quotients x^3).

[1] (e) Evaluate $\binom{1/3}{3}$.

By definition, $\binom{1/3}{3} = \frac{1/3\cdot(1/3-1)\cdot(1/3-2)}{3\cdot2\cdot1} = \frac{5}{81}$.

[4] (f) Find the coefficient of x^{26} in the following function: $x^3(x^3 + x^5 + x^7 + x^9 + \cdots)^5$.

Let the function be denoted $f(x)$. We simplify:

$$\begin{aligned} f(x) &= x^3x^{15}(1 + x^2 + x^4 + x^6 + \cdots)^5 \\ &= x^{18}(1-x^2)^{-5} \quad \text{(geometric series)} \\ &= x^{18}\sum_{k=0}^{\infty}\binom{-5}{k}(-1)^k x^{2k} \quad \text{(generalized binomial theorem)} \end{aligned}$$

As $26 = 18 + 2\cdot 4$, we see that

$$[x^{26}]f(x) = (-1)^4\binom{-5}{4} = \binom{4+5-1}{4} = \binom{8}{4} = 70\,.$$

[10] **5.** A group of 21 people needs to split into four groups during a trip, one group will continue by foot, another by tandem bikes, the third by canoe, and the fourth by kayak. They need to respect the following

- the number of people that go by canoes is even and at least 2;
- the number of people that go by kayaks is even;
- the number of people that go by tandem bikes is even and at least 4.
- the number of people that go by foot is at least 2 and at most 6

Suppose we only care about how many canoes, kayaks and bikes will be used (each canoe, kayak and tandem bike will cary exactly two people). How many different possibilities are there? Use generating functions.

Let a, b, c, and d denote the number of people travelling by canoes, kayaks, tandem bikes, and by foot respectively. The imposed conditions are

- $a+b+c+d=21$,
- a, b, c are even,
- $a \geq 2$,
- $c \geq 4$, and
- $2 \leq d \leq 6$.

Thus, the required number is the coefficient of x^{21} in

$$\begin{aligned} f(x) &= (x^2+x^4+x^6+\cdots)(1+x^2+x^4+\cdots)(x^4+x^6+x^8+\cdots)(x^2+x^3+x^4+x^5+x^6) \\ &= x^8(1+x+x^2+x^3+x^4)(1+x^2+x^4+x^6+\cdots)^3 \\ &= x^8(1+x+x^2+x^3+x^4)(1-x^2)^{-3} \end{aligned}$$

Again we use the generalized binomial theorem to find that

$$\begin{aligned} [x^{21}]f(x) &= [x^{12}](1-x^2)^{-3} + [x^{10}](1-x^2)^{-3} = \binom{-3}{6} - \binom{-3}{5} \\ &= \binom{3+6-1}{6} + \binom{3+5-1}{5} = \binom{8}{2} + \binom{7}{2} = 28+21 = 49\,. \end{aligned}$$