Name: ________________________

Signature: ________________________

Final Examination
MACM 201
2001-2

7 August 2001
12:00–15:00

Instructions:

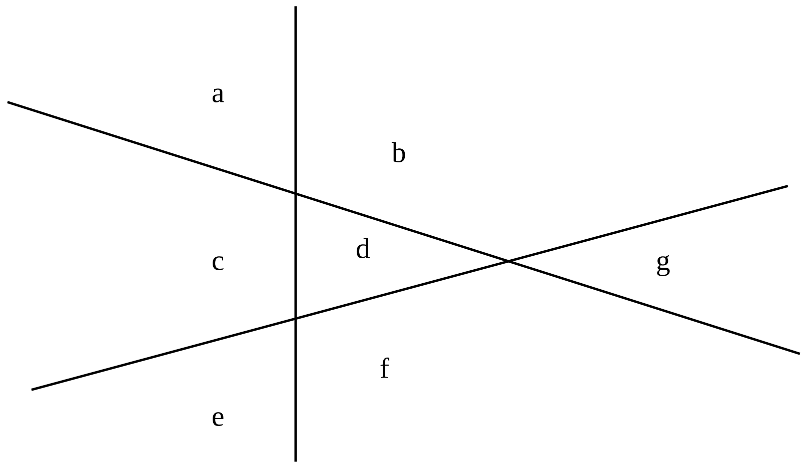
1. *Do not open this booklet until told to do so.*
2. *Write your name above in block letters and sign below your name. Write your student number and your family name in the boxes on the inside of the back cover page.*
3. *This exam has 15 questions and 17 pages. Please check to make sure your exam is complete.*
4. *Each question is worth 10 marks. Attempt all questions.*
5. *You are allowed to use one $8\frac{1}{2} \times 11''$ sheet of notes (double sided). No other notes, books or computing devices may be used.*
6. *You may lose points if your solution is not properly justified.*
7. *If the space provided for your solution is insufficient you may use the back of the previous page.*
8. *Good luck!*

1. *(10 marks)*
Let $P(n)$ denote the proposition:
"Using only 3-cent and 4-cent stamps, one can make up a total of n cents."

Use mathematical induction to prove that $P(n)$ is true for all $n \geq 6$.

2.

Let n be a positive integer and let R_n denote the number of regions into which n lines divide the plane, if we assume that each line intersects the remaining $n-1$ lines in $n-1$ distinct points. The following illustration of the case $n=3$ shows that $R_3=7$ (the seven regions are denoted by a,b,c,d,e,f,g).



a) *(5 marks)*

Find a recurrence relation and initial condition(s) for the sequence R_n. (Hint: How many regions are bisected when one draws the n-th line?)

b) *(5 marks)*

Use the answer to part (a) to prove that R_n is $O(n^2)$. Hint: What is the general form of a solution to the recurrence relation that you found in part (a)?

3.

a) *(5 marks)*

Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with the initial conditions $a_0 = 0$, $a_1 = 1$.

b) *(5 marks)*

Find the general form of a solution to the recurrence relation

$$a_n = a_{n-1} + 5a_{n-2} + 3a_{n-3}.$$

Hint: $x^3 - x^2 - 5x - 3 = (x+1)^2(x-3)$.

4.
A snack at a party will consist of a basket of 12 pieces of fruit, each of which is an apple, an orange or a banana. Pieces of the same fruit are indistinguishable and the ordering of the fruits in the basket does not matter.

Let B denote the number of ways in which the basket can be composed.

a) *(5 marks)*
Find a function $G(x)$ such that B is the coefficient of x^{12} in $G(x)$. Explain how you arrived at your answer.

b) *(5 marks)*
Find the value of B.

5. *(10 marks)*
Let A, B, C, D be four sets such that each set contains exactly 9 elements, every two different sets have exactly 2 elements in common and no element belongs to more than two sets. Find the number of elements of $A \cup B \cup C \cup D$.

6. *(10 marks)*
Mark each of the following statements as true (**T**) or false (**F**) in the box placed next to the statement. You will receive one mark for each correct answer, no mark for no answer and lose one mark for each incorrect answer.

The word 'iff' means "if and only if."

a) ☐ The relation $\subseteq$ is a total (linear) order on the set of all subsets of $\{1, 2, 3, 4\}$.

b) ☐ There exists an equivalence relation which is antisymmetric.

c) ☐ If R is a transitive relation then $R^2 \subseteq R$.

d) ☐ For every relation R the following holds:
If $R^2 \subseteq R$ and $R^3 \subseteq R$, then R is transitive.

e) ☐ The total number of all relations on the set $\{1, 2, 3, 4\}$ is $2^{\binom{4}{2}}$.

f) ☐ For any two relations R and S we have $R \circ S = S \circ R$.

g) ☐ The relation R on the set of all ordered pairs of integers defined by $(a, b)R(c, d)$ iff $a + d = b + c$ is an equivalence relation.

h) ☐ The relation R on the set of all rational numbers defined by aRb iff $|a - b| \leq 1$ is an equivalence relation.

i) ☐ The relation $\simeq$ on the set of all simple graphs defined by $G \simeq H$ iff G and H are isomorphic is an equivalence relation.

k) ☐ For any set A and any relation R on A the following holds:
If (A, R) is a poset then (A, R^{-1}) is a poset.
Note: R^{-1} denotes the inverse of R, i.e. $aR^{-1}b$ iff bRa.

7. *(10 marks)*
Let R be an equivalence relation on a set A. For each $a \in A$ let $[a]$ denote the equivalence class of a, that is, the set of all those elements z of A such that aRz. Assume that for two particular elements $x, y \in A$ we have xRy. Prove that under this assumption it follows that $[x] = [y]$.

8. *(10 marks)*
Draw the Hasse Diagram for the poset (A, R) where $A = \{2, 3, 5, 6, 15, 18, 20, 40\}$ and xRy means that x divides y.

9. *(10 marks)*
Draw all non-isomorphic simple graphs with 5 vertices and 5 edges. (Hint: There are exactly six of them.)

10. *(10 marks)*
A simple circuit in a graph G is a circuit that contains every edge of G at most once (but a vertex can appear any number of times). The length of a simple circuit is the number of edges in the circuit.

Find the maximum possible length of a simple circuit in the complete bipartite graph $K_{n,n}$. Give your answer in the form of a simple expression in the variable n. Hint: There are two cases to distinguish.

Give a brief explanation of your solution.

11.

a) *(5 marks)*

Find a planar simple graph with 5 vertices and 9 edges or prove that such a graph cannot exist.

b) *(5 marks)*

Find a connected simple graph with 7 vertices and 11 edges which is both bipartite and planar, or prove that such a graph cannot exist.

12. *(10 marks)*
Find a simple graph G such that the chromatic number of G is 3 and G does not contain a subgraph isomorphic to K_3. Show a coloring of G with 3 colors. (Use the symbols A, B, C to denote the three colors.)

13.

a) *(5 marks)*
Give the definition of a tree.

Let T be a tree, and let u and v be two vertices of T. Prove that there is a unique simple path between u and v in T.

b) *(5 marks)*
Explain briefly why, in general, it is not possible to sort three numbers using only two binary comparisons.

Use a similar argument to prove that, in general, <u>four</u> binary comparisons are not sufficient to sort <u>four</u> numbers. (Hint: What is the maximum number of leaves in a binary decision tree of height h?)

14. *(10 marks)*
Let s denote the following sequence of 8 letters:

$$DCABDDAD$$

If one uses a code that assigns to each of the four letters A, B, C, D a unique string of two bits (i.e. 00, 01, 10 or 11), then a total of $8 \cdot 2 = 16$ bits are needed to encode the given sequence s.

Construct a binary prefix code P such that fewer than 16 bits are needed to encode s if one uses the code P. Compute the number of bits needed to encode s if one uses the code P.

15. *(10 marks)*
Find the postfix form of the expression $(a * (b + c)) + (((d - 1)/e) \uparrow 3)$.

Last Name	
Student number	

Question	Maximum	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
Total	150	