Macm 201 Spring 2001

Final exam

Name___________________________

Student Number___________________

Signature_________________________

Each of the 22 questions counts 10 points. On most, but not all, you will receive 0, 5, or 10 points. Show your work, even if the question just asks for a number, in order to receive partial credit. On each question, please clearly indicate your final answer.

Calculators are allowed, as is a single 8 1/2 x 11 page of notes (both sides). Please stop promptly at 11:30 am.

Good luck!

1	6	11	16	21
2	7	12	17	22
3	8	13	18	Total
4	9	14	19	
5	10	15	20	

1. For each $n \geq 1$, let $S(n) = \sum_{\emptyset \neq \{a_1,\cdots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 a_3 \cdots a_k}$, where the summation is over all of the non-empty subsets $\{a_1, a_2, a_3, \cdots, a_k\}$ of {1,2,3,…,n}. For example, when $n = 3$, $S(3) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3}$. Show that for each $n \geq 1$, if $S(n) = n$ then $S(n+1) = n+1$. Your argument must work for an arbitrary n, not for just a particular value of n. (Note that since $S(1) = 1$, it follows by induction that $S(n) = n$ for all $n \geq 1$.)

2. Find a recurrence relation for the sequence $\{a_n\}$, where for each n • 1, a_n denotes the number of bitstrings of length n which do contain two consecutive 0’s. For example, $a_3 = 3$, since the bitstrings of length 3 which contain two consecutive 0’s are the bitstrings 000, 001, and 100. Of course, $a_1 = 0$, and $a_2 = 1$. You do not need to prove that your recurrence relation is correct, but your scratch work should give some idea of how you found it. (You can check your recurrence relation against $a_3 = 3$ as a check, and against $a_4 = 8$.) Circle your final answer

3. Solve the recurrence relation $(a_{n+2})^2 - 5(a_{n+1})^2 + 4(a_n)^2 = 0$, $n \geq 0$, $a_0 = 0$, $a_1 = 3$.

(Hint: Set $b_n = (a_n)^2$.) Circle your final answer.

4. (a) Find the simplest form (this is defined below) for the generating function for the sequence $\{a_n\}$, where for each $n \bullet 0$, a_n is the number of ways to distribute n (identical) oranges to 100 (distinct) children, where each child must receive at least two oranges. By the simplest form is meant an expression for a_n which does not contain any infinite summations. Circle your final answer.

(b) Find the number of ways that 500 (identical) oranges can be distributed among 100 (distinct) children, where each child must receive at least two oranges. Express your answer in terms of a binomial coefficient. Circle your final answer.

5. (a) How many elements are in the union of four sets if each of the sets has 100 elements, each pair of the sets shares 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets? Express your answer as a number. Circle your final answer.

(b) How many onto functions are there from the set X = {a,b,c,d,e} to the set Y = {1,2,3}? (Recall that a function f from X to Y is an onto function if there exist p, q, r in X such that f(p) = 1, f(q) = 2, f(r) = 3.) Express your answer as a number. Circle your final answer.

6. Let D be the following relation on the set Z of all integers (positive, negative, and zero): for each pair of integers x and y, x D y if and only if x divides y. For example, 5D20, 2D(-4) and (-3)D12. Which two of the following statements are true? Note: exactly two are true. (Circle the true statements, and circle at most two!)

1. D is symmetric and D is anti-symmetric.
2. D is not symmetric and D is not antisymmetric.
3. D is not symmetric and D is anti-symmetric.
4. D is symmetric and D is not anti-symmetric.
5. There exists at least one x in Z such that: (xDy is false for all y in Z).
6. There exists at least one x in Z such that: (xDy is true for all y in Z).

7. Let R be an equivalence relation on the set A. For each x in A, let [x] denote the equivalence class which contains x. That is, [x] is the set of all those elements z in A such that xRz. Assume that for a particular two elements x and y, we have $[x] \cap [y] \neq \varnothing$, that is, $[x] \cap [y]$ is not empty. Use the assumption that $[x] \cap [y] \neq \varnothing$ to prove that $[x] = [y]$.

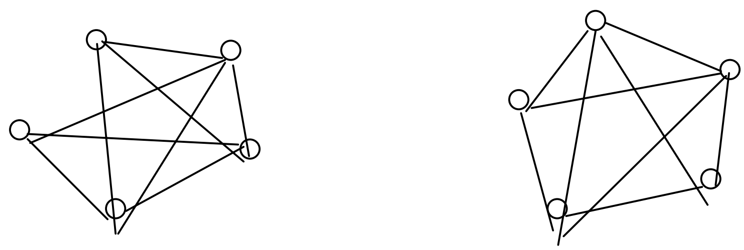
8. Recall that a <u>well-ordered set</u> is a partially ordered set (S, •) which has two properties: 1. The order relation • is a total ordering (this means that for every two elements x, y of S, either x • y or y • x or both). 2. Every nonempty subset of S has a least element with respect to the order relation •.

Let S be the set of all positive real numbers, and let • denote the ordinary "less than or equal to" order relation on S. Is (S, •) a well-ordered set? Explain.

9. Give an example of a simple graph which has exactly 100 vertices, exactly 2500 edges, and which has chromatic number 2.

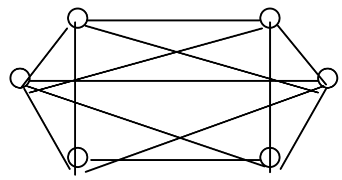
10. How many distinct simple graphs are there on the vertex set $V = \{1,2,3,4,5\}$? Circle your final answer. (We're not interested in isomorphism here.)

11. Are these two graphs isomorphic? Explain.

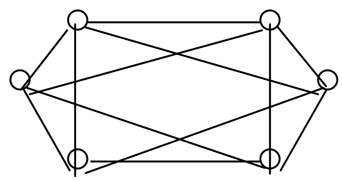


12. Let G be any simple connected graph which has exactly 8 vertices with odd degrees. What is the minimum number k such that there exist paths $P_1, P_2, P_3, \cdots, P_k$ in G such that each edge of G belongs to exactly one of these paths? Explain. (Your explanation has to apply to all possible simple graphs that satisfy the given condition.) Circle your final answer.

13. (a) Show, by using a suitable corollary of Euler's Formula, that the following graph is non-planar.

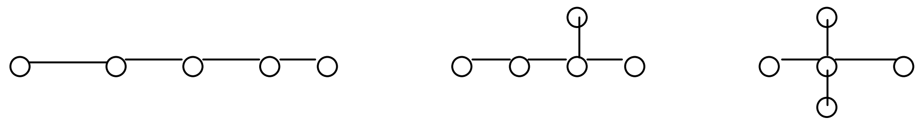


(b) Is the following graph planar? If so, draw it in the plane (with no edge crossings). If not, give a good reason. (The reason "I couldn't draw it in the plane" is not a good reason!)

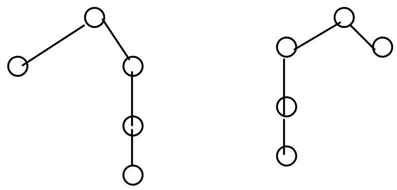


14. Let G be any simple planar graph with 5 connected components, 32 edges, and 22 vertices. If G is drawn in the plane with no edge crossings, how many regions will there be? (Hint: use Euler's Formula on each connected component of G.) Your answer should be a number. Circle your final answer.

15. There are 3 pairwise non-isomorphic trees on 5 vertices, namely:



How many pairwise non-isomorphic rooted trees are there on 5 vertices? (These rooted trees are not ordered. That is, the rooted trees



are isomorphic.) Circle your final answer.

16. Let T be a full binary tree which has i internal vertices, l leaves, and n vertices altogether. Then we know that $n = 2i + 1$. Explain why it follows that $l = i + 1$.

17. Write down the postfix form (sometimes called the "reverse Polish notation") for the arithmetic expression $(a * b + c * d) - 3$. Circle your final answer. (In more ordinary notation, the expression is $(ab + cd)^3$.)

18. Using the Merge Sort algorithm, how many comparisons are needed to merge the two sorted lists 4,6,7,8,9 and1,2,3,5? Circle your final answer. _____

19. There are altogether $2^8 = 256$ distinct Boolean functions $F(x, y, z)$ on three Boolean variables. How many of these satisfy the identity $F(x, y, z) = F(x+y, y+z, z+x)$ for all x, y, z? Your answer should be a number. Circle your final answer.

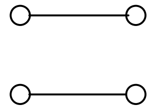
20. Consider the Boolean function F(x,y,z) given by the following table:

x	y	z	$F(x,y,z)$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	1

We know that the "sum-of-products expansion" of this function is $F(x,y,z) = xyz + xy\overline{z} + x\overline{y}z + \overline{x}yz + \overline{x}\,\overline{y}\,\overline{z}$, and this is obtained by looking at the triples (x,y,z) such that F(x,y,z) = 1. By looking at the triples (x,y,z) such that F(x,y,z) = 0, find the "product-of-sums expansion" of this function.

21. In a simple circuit in a graph, each edge of the graph appears at most once, but a vertex may appear any number of times. Find the maximum possible number of edges in a simple circuit in the complete graph K_{2n}. Your answer should be a simple expression involving n. Circle your final answer. You don't need to give a complete proof that your answer is correct, but give a very brief explanation.

22. If G = (V, E) is a simple undirected graph, a <u>perfect matching</u> in G is a spanning subgraph H (H contains al the vertices of G) such that every vertex in H has degree 1. For example, the graph



Is a perfect matching of the complete graph K_4. In fact, if we give the complete graph K_4 the vertex set V = {1,2,3,4}, then K_4 has exactly three <u>distinct</u> perfect matchings. These are M_1 = { {1,2}, {3,4} }, M_2 = { {1,3}, {2,4} }, and M_3 = { {1,4}, {2,3} }. Also, by counting carefully, we see that the complete graph K_6 has exactly 15 distinct perfect matchings. How many perfect matchings does the complete graph K_{2n} have?

A $\binom{2n}{2}$, if n is odd, $\frac{1}{2}\binom{2n}{2}$ if n is even.

B $\frac{(2n)!}{n!2^n}$.

C $4^{n-1}-1$.

D none of the above.

Give a <u>brief</u> explanation. (Hint: Carefully count the perfect matchings of K_6.