Macm 201 Fall 2000

Final exam

Name____________________________

Student Number___________________

Signature_________________________

Each of the 23 questions counts 10 points. Calculators are allowed, as is a single 8 1/2 x 11 page of notes. Good luck!

1	6	11	16	21
2	7	12	17	22
3	8	13	18	23
4	9	14	19	Total
5	10	15	20	

1. Prove by mathematical induction that $2^n > n^2$, $n \geq 5$.

2. For $n \geq 1$, let a_n be the number of bitstrings of length n which do not contain three consecutive 0's. (Thus for example $a_1 = 2$, $a_2 = 4$, $a_3 = 7$.) Find a recurrence relation for the sequence $\{a_n\}$. (Do not solve it!) (Hint: See if your solution gives the right value for a_4.)

3. Write down the general form of the solution of the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$, (Hint: $(x-1)^3 = x^3 - 3x^2 + 3x - 1$.)

4. (5 points) Circle the correct answer. Let $f(x) = x^{100}, g(x) = 2^x$. Then

A $f(x)$ is $O(g(x))$ and $g(x)$ is not $O(f(x))$

B $f(x)$ is not $O(g(x))$ and $g(x)$ is $O(f(x))$

C $f(x)$ is not $O(g(x))$ and $g(x)$ is not $O(f(x))$

D $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$

(5 points) Circle the correct answer. On the positive integers, define the two functions f, g in the following way:

f(n) = 1 when n is even, f(n) = n when n is odd.

g(n) = n when n is even, g(n) = 1 when n is odd.

A $f(x)$ is $O(g(x))$ and $g(x)$ is not $O(f(x))$

B $f(x)$ is not $O(g(x))$ and $g(x)$ is $O(f(x))$

C $f(x)$ is not $O(g(x))$ and $g(x)$ is not $O(f(x))$

D $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$

5. Give a closed form generating function for the sequence 2, 0, 6, 0, 18, 0, 54, 0, 162, 0, … . (This means: $2 + 6x^2 + 18\,x^4 + 54x^6 + \ldots$ is not an acceptable answer.) (Hint: use the identity $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$.)

6. Let A, B, C, D be finite sets. Write down the inclusion-exclusion formula for $|A \cup B \cup C \cup D|$.

7. Let A be a set with 3 elements.

(a) (5 points) How many binary relations are there on the set A? (Hint: the answer is one of 1, 2^2, 2^3, 2^4, … , 2^{512}.)

(b) How many equivalence relations are there on the set A?

8. Define the relation R on the set of real numbers as follows: For every two real numbers x, y, xRy if and only if the number xy is <u>rational</u>. Prove that R is an equivalence relation, or prove that R is not an equivalence relation.

9. Draw the Hasse diagram of the partial ordering R = {(a, b) | a divides b} on the set {1,2,3,4,6,12}. That is, A = {1,2,3,4,6,12}, and R is the relation on A defined by: for all x, y in A, xRy if and only if x is a divisor of y.

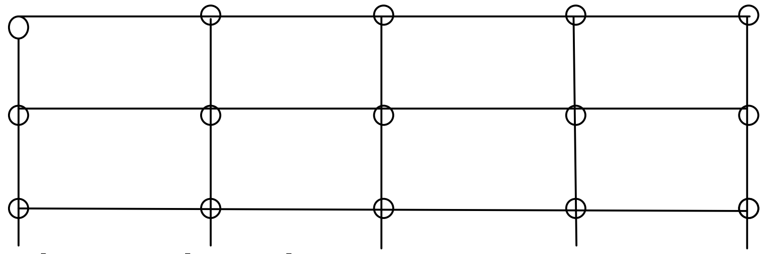
10. (a) (5 points) Circle the correct answer. How many non-isomorphic undirected simple graphs G are there such that G has exactly 6 vertices, and every vertex has degree two? (G has no loops and no multiple edges.)

A 1 B 2 C 30 D 70 E 720

(b) (5 points) Circle the correct answer. Given a specific vertex set V with six elements, say V = {1,2,3,4,5,6}, how many distinct undirected simple graphs G = (V, E) (no loops and no multiple edges) are there, such that G has vertex set V, and every vertex has degree two? (G has no loops and no multiple edges.

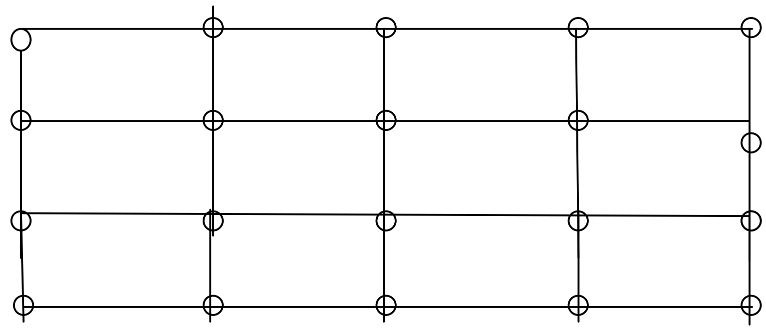
A 1 B 2 C 30 D 70 E 720

11. (a) (5 points) Circle the correct answer. Let G be the following graph



A G has a Hamilton cycle.
B G has a Hamilton path, but no Hamilton cycle.
C G does not have a Hamilton path.

(b) (5 points) Circle the correct answer. Let H be the following graph.



A H has a Hamilton cycle.
B H has a Hamilton path, but no Hamilton cycle.
C H does not have a Hamilton path.

12. (a) (5 points) Draw a simple planar graph, with six vertices, in which each vertex has degree 3.
(b) (5 points) Draw a simple non-planar graph, with six vertices, in which each vertex has degree 3.

13. Using a binary tree, make up a binary prefix code for the letters a,b,c,d,e. (There are many different correct answers to this question.)

14. Find the postfix form of the expression $((x+y)\uparrow 2)+((x-4)/3)$.

15. Let G be the simple complete graph on the vertex set V = {1,2,3,4,5}. (That is, G has 10 edges, and V(G) = {1,2,3,4,5}.) Assign to each edge {i,j} the weight i+j. (Thus the edge {1,2} has weight 3, the edge {3,5} has weight 8, and so on.) Draw a minimum spanning tree of G; be sure to label the vertices with 1,2,3,4,5.

16. (a) (5 points) How many Boolean functions are there on 10 Boolean variables?

(b) (5 points) How many relations are there on a set with 10 elements which are simultaneously reflexive and symmetric?

17. Prove the Boolean identity $x + xy = x$.

18. Find the sum-of-products expansion of the function $F(x,y,z) = (x+y)z$.

19. Define the Boolean operator | (the NAND operator) by
$1|1 = 0$ and $1|0 = 0|1 = 0|0 = 1$. Show that $xy = (x|y)|(y|x)$.

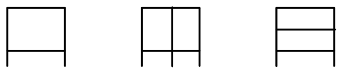
20. (a) (5 points) Circle the correct answer. What is the coefficient of x^8 in the power series $\frac{1}{(1-x)(1-2x)}$?

A $3^8 - 1$ B 7 C $2^8 + 1$ D $2^9 - 1$ E 64

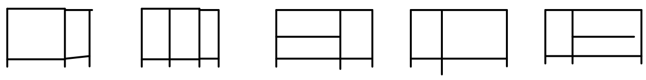
(b) (5 points) Circle the correct answer. Let G be a simple connected planar graph. Assume that for some planar embedding of G (drawing of G in the plane with no edge-crossings) there are 5 regions, and every region (including the infinite region) has exactly 6 edges in its boundary. (That is, every region has degree 6.) Then the number of vertices of G is:

A 10 B 12 C 30 D 9 E cannot be determined from the given information.

21. In how many ways can a 2 x n board be tiled using 1 x 2 and 2 x 2 pieces? ("Tiled" means covered with non-overlapping pieces.) For example, when n = 2 there are three ways:



When n = 3 there are 5 ways:



22. Suppose G is a simple graph on 6 vertices. Show that G cannot be isomorphic to its complement.

23. Prove that $x + xy + xyz + xyzw = x$ for all possible values of the Boolean variables x,y,z,w. (Hint: There is a very short proof.)