SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Final Exam - Solutions

MACM 201 Spring 2008

Instructor: Robert Šámal

April 14, 2008, 12:00 – 3:00

1. A group of 25 students is deciding which courses each of them will take. Suppose that there are only 4 courses available, each student may enroll in any number of them, including 0. (Note: courses are distinguishable, students are distinguishable!)

[3] (a) How many possibilities are there?

Every student has 2^4 possibilities. So altogether there are $(2^4)^{25} = 2^{100}$ possibilities.

[7] (b) How many of the possibilities have at least one student enrolled in each of the 4 courses?

An easy way: *Each course has 2^{25} possibilities for the list of students enrolled, exactly one of which is "bad" — namely when the set is empty, no student enrolled. The choices for different courses are independent, thus we have $(2^{25}-1)^4$ possibilities.*

The standard way: *Let S be the set of all "enrollments", c_i the condition that the i-th course gets no students. We will use PIE.*

- $N = (2^4)^{25}$ *(from part (a))*
- $N(c_1) = N(c_2) = N(c_3) = N(c_4) = (2^3)^{25}$*: we have one course that is "empty", so no student chose it. That means that each student has 2^3 possibilities to choose the courses.*
- $N(c_1c_2) = \cdots = N(c_3c_4) = (2^2)^{25}$*: for similar reason*
- $N(c_1c_2c_3) = \cdots = N(c_2c_3c_4) = (2^1)^{25}$.
- $N(c_1c_2c_3c_4) = (2^0)^{25} = 1$.
- *Altogether we got*
$$S_k = \binom{4}{k} 2^{25(4-k)}$$
.
- *This yields the final answer*
$$\sum_{k=0}^{4} (-1)^k \binom{4}{k} 2^{25(4-k)}.$$

(You may want to check that this gives the same anwer as the first approach.)

[10] **2.** Ten families (each with five members—mother, father and three children) are going to visit a cinema. In how many ways can they form a queue if no family is standing "together" (that is its members don't occupy five consecutive places in the queue)?

Let S be the set of all possible ways to arrange the 50 people in a line, let c_i be the condition that the i-th family stands "together".

- $|S| = 50!$ *(permutations of 50 objects)*
- $N(c_i) = (50 - 5 + 1)!5!$
  We do the usual thing: if the i-the family stand together, we may consider it as one "compound object". So we are looking at permutations of $50 - 5$ people and 1 group, altogether $50 - 5 + 1$ objects. We also need to count the permutations within the i-th family.
- *Similarly, we get $N(c_{i_1} \cdots c_{i_k}) = (50 - 4k)!(5!)^k$. (We may check that it gives the correct answer for $k = 0, 1$.)*
- *This yields $S_k = \binom{10}{k}(50 - 4k)!(5!)^k$.*
- *So, the final answer is*

$$\sum_{k=0}^{10} (-1)^k \binom{10}{k} (50 - 4k)!(5!)^k.$$

3. Ten families (each with five members) are going to have a dinner (after seeing the movie in question 2 :–)). Each family will be occupying one table for five (there is exactly ten of them available). Each such table has five chairs, each of a different color (red, green, blue, white, and yellow). So everybody's position is described by the number of the table and the color of the chair.
You don't need to provide numerical answers, an (exact) formula involving factorials is just fine.

[2] (a) How many different seating arrangements are there?

We have $10!$ *ways how to assign tables to families,* $5!$ *ways how to let the members of one family choose their chairs. Thus, the total number is*

$$10!(5!)^{10}.$$

[4] (b) Suppose that the people decided to change the seating before they eat the dessert (to enjoy a view from another window). That is, each family moves to a different table, and everybody chooses some chair. How many possibilities are there for this new seating arrangement?

We have d_{10} *ways how to assign new tables to families,* $5!$ *ways how to let the members of one family choose their chairs. Thus, the total number is*

$$d_{10}(5!)^{10}.$$

[4] (c) The same question as in the previous part, but now everybody wants to sit on a different-colored chair after the change. How many possibilities are there this time?

We have d_{10} *ways how to assign new tables to families,* d_5 *ways how to let the members of one family choose their chairs (they need different color than before!). Thus, the total number is*

$$d_{10}(d_5)^{10}.$$

In parts (b), (c) we have used d_n *to mean the number of derangements (permutations without fixed points) of* n *objects. We know from class that*

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

So we may plug this in the above formulas to get the desired answer expressed using factorials. (You didn't have to really plug this in, but mentioning the formula for d_n *was required.)*

[10] **4.** Solve the following recurrence relation

$$a_{n+2} - 4a_{n+1} + 3a_n = n \quad (n \geq 0), \qquad a_0 = 1, a_1 = 1\,.$$

The characteristic equation is $t^2 - 4t + 3 = 0$, its roots are $t_1 = 3$, $t_2 = 1$. So, the general homogeneous solution is of form $a_n^{(h)} = A3^n + B1^n$. We will look for a particular solution in the form $a_n^{(p)} = Cn^2 + Dn$ (C, D suitable constants). Observe that we "multiply by n", as part of the homogeneous solution is a constant. Plugging in the recurrence relation we obtain

$$\begin{aligned} C(n+2)^2 + D(n+2) - 4(C(n+1)^2 + D(n+1)) + 3(Cn^2 + Dn) = n \\ 0 \cdot n^2 + (4C + D - 8C - 4D + 3D) \cdot n + (4C + 2D - 4C - 4D) = n \end{aligned}$$

A simple (in fact, the only) way how to satify this equation for every n is solving a system of two equations

$$\begin{aligned} n(-4C) &= 2n \\ -2D &= 0 \end{aligned}$$

This gives us $C = -1/4, D = 0$. We've found $a_n^{(p)} = -n^2/4$. So the general solution is $a_n = a_n^{(h)} + a_n^{(p)} = A3^n + B - n^2/4$. To determine A and B we put $n = 0$ and $n = 1$:

$$\begin{aligned} 1 &= a_0 = A + B - 0 \\ 1 &= a_1 = 3A + B - 1/4 \end{aligned}$$

this gives $A = 1/8$ and $B = 7/8$.

Altogether, we found that

$$a_n = \frac{1}{8}3^n + \frac{7}{8} - \frac{1}{4}n^2 \qquad (n \geq 0).$$

[10] **5.** Find the number of solutions to

$$a + 3b + 6c = 211\,,$$

so that a, b, c are integers, $a \geq 3$, $b \geq 2$, and $c \geq 1$. Use generating functions.

We use the usual "combinatorial property of multiplication", the first paranthesis will contain terms x^a, the second x^{2b}, and the thirst x^{3c}. So we are interested in the following quantity

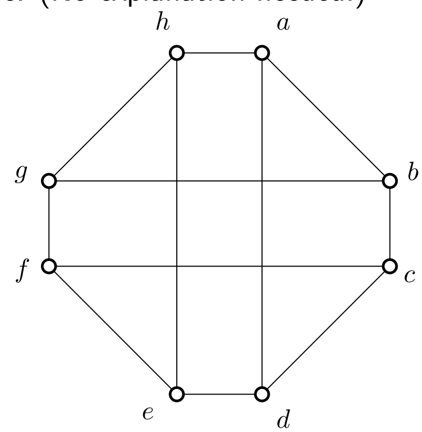
$$\begin{aligned}
&[x^{211}](x^3 + x^4 + \cdots)(x^{3\cdot 2} + x^{3\cdot 3} + x^{3\cdot 4} + \cdots)(x^{6\cdot 1} + x^{6\cdot 2} + x^{6\cdot 3} + \cdots) \\
&= [x^{211}]\frac{x^3}{1-x} \cdot \frac{x^6}{1-x^3} \cdot \frac{x^6}{1-x^6} \\
&= [x^{211}]\frac{x^{15}(1 + x + x^2 + \cdots + x^5)(1 + x^3)}{(1-x^6)^3} \\
&= [x^{196}]\frac{\cdots + 2x^4 + \cdots}{(1-x^6)^3} \\
&= 2[x^{192}]\frac{1}{(1-x^6)^3}
\end{aligned}$$

(note that $192 = 32 \cdot 6$*)*

$$\begin{aligned}
&= 2\binom{-3}{32} \\
&= 2\binom{34}{2} (= 1122)
\end{aligned}$$

(We have used the binomial theorem and standard methods to modify a generating function. The numerical answer was optional.)

6. Answer the following questions for the graph in the figure. (No explanation needed.)



[1] (a) How many vertices does the graph have?

8

[1] (b) How many edges does the graph have?

12

[1] (c) If the graph was drawn without crossings, how many faces would it have?

6

[1] (d) Does the graph contain a Hamilton cycle?

Yes.

[1] (e) Does the graph contain an Euler trail?

No.

[2] (f) How many subgraphs contain all outside edges, that is, all of the edges $ab, bc, cd, de, ef, fg, gh$, and ha.

$2^4 = 16$

[2] (g) What is the minimum number of edges in a connected spanning subgraph?

$8 - 1 = 7$

[1] (h) What is the distance (length of a shortest path) between a and g?

2

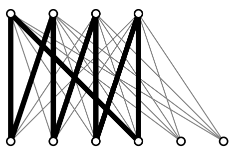
7. Decide, what is the maximal length of a

1. cycle
2. circuit

in the following graphs. Explain your results!

[5] (a) $K_{4,6}$

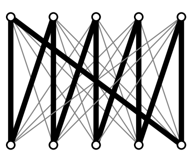
1. Every second vertex on a cycle in $K_{4,6}$ contains vertex from the part with 4 vertices. This means there can be at most **8** *vertices on a cycle. This is possible:*



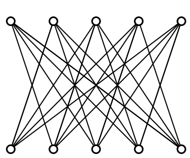
2. All vertices have degrees 4 or 6, and the graph is connected. Thus, there is an eulerian circuit. (And we don't need to actually draw it.) This means the desired length is **24**.

[5] (b) $K_{5,5}$

1. The cycle obviously cannot have more than **10** *vertices, and this is possible:*

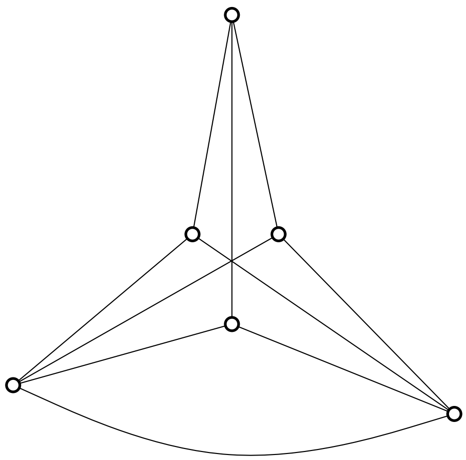


2. Equivalently, we need to find a subgraph with eulerian circuit and as many edges as possible. As there are 10 vertices of odd degree, we need to remove at least $10/2 = 5$ edges to get graph with eulerian circuit, and this is indeed possible, if we remove five edges that do not share a vertex:

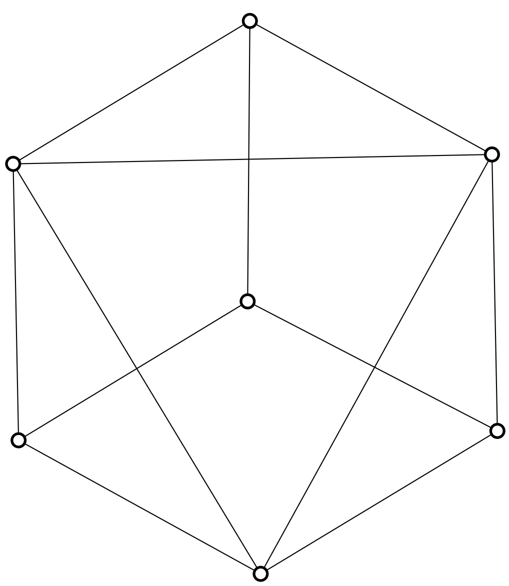


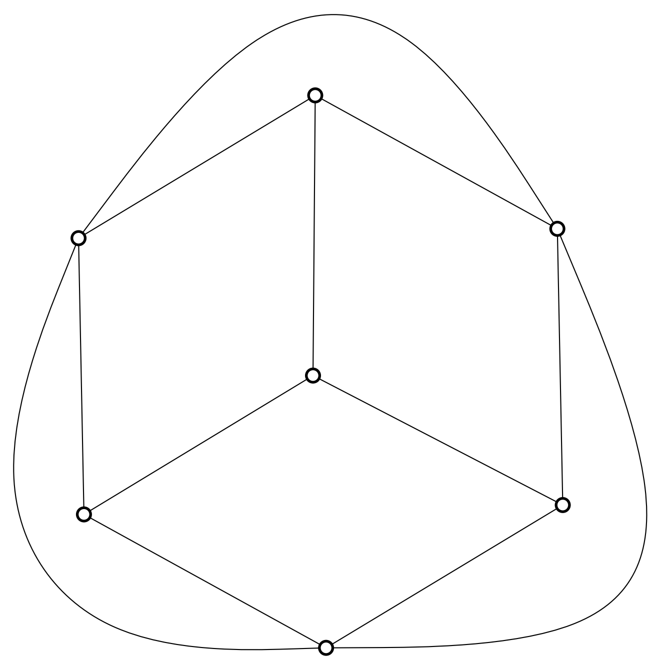
This yields a graph with $5^2 - 5 = 20$ edges, so **20** *is the length of the longest circuit in our graph.*

[10] **8.** Are the following graphs planar? If yes, draw the graph without crossings, if not, explain why not. [2 marks for each graph]

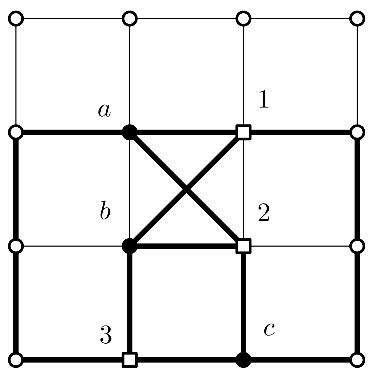


This is actually the $K_{3,3}$ plus one edge — so it is not planar.

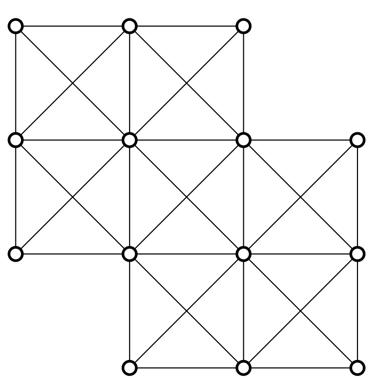




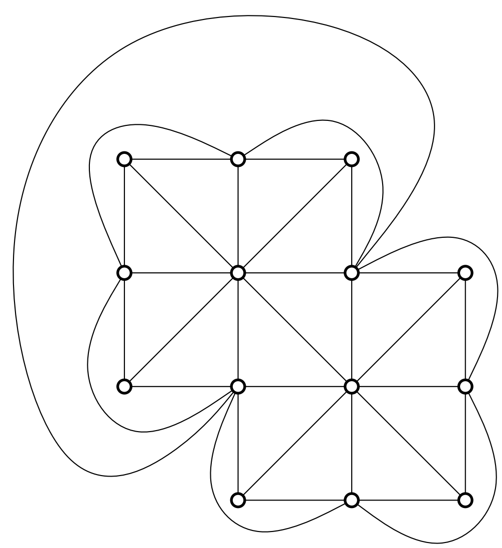
This is a planar graph.

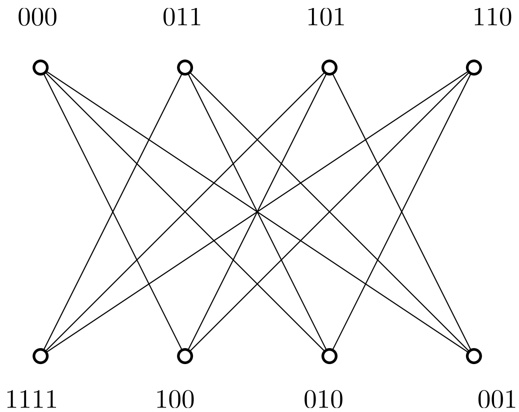


Not planar — a subdivision of $K_{3,3}$ is provided.

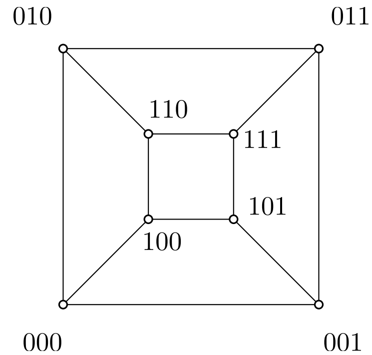


This is a planar graph.

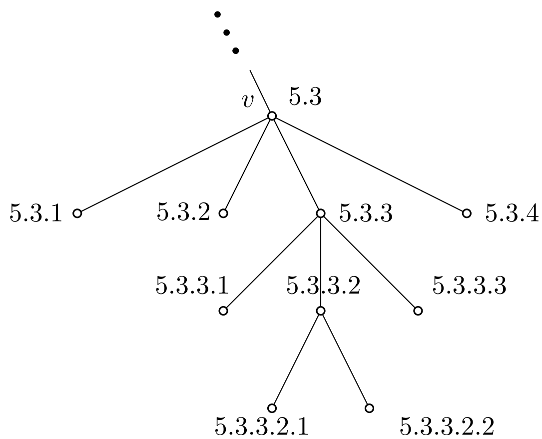




Believe it or not, this is the cube.



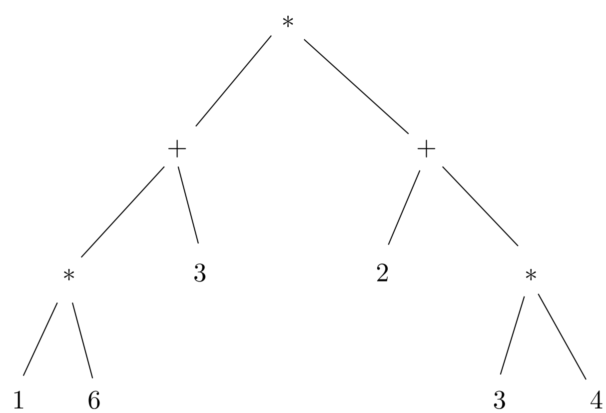
[4] **9.** (a) In the figure is *a part of* a rooted tree. (However, for each vertex in the figure all of its children are shown.) You are given the label of one of the vertices according to the universal address system. Determine the labels of all the other vertices in the figure. What is the level of vertex v?



Vertex v is at level 2.

[2] (b) Construct the binary rooted tree for the following arithmetic expression:

$$(1 * 6 + 3) * (2 + 3 * 4)$$



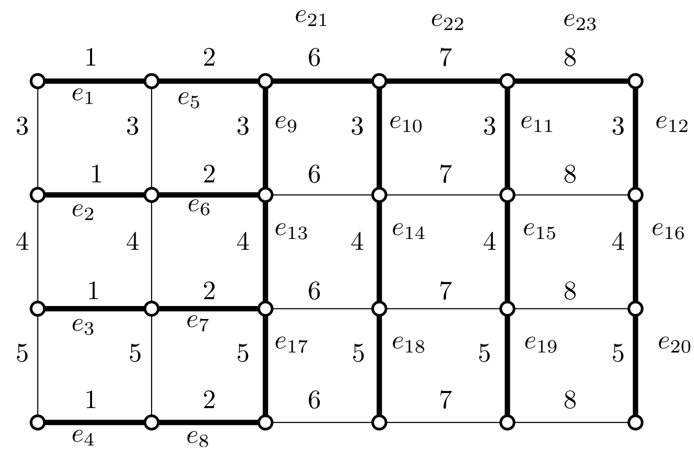
[2] (c) Use the postorder traversal to write the expression in postfix notation.

$1\ 6\ *\ 3\ +\ 2\ 3\ 4\ *\ +\ *$

[2] (d) Use the preorder traversal to write the expression in prefix notation.

$*\ +\ *\ 1\ 6\ 3\ +\ 2\ *\ 3\ 4$

10. We will be considering spanning trees in the following graph, with weights as given in the figure.



[1] (a) How many edges can a spanning tree of this graph have?

The graph has $4 \cdot 6 = 24$ vertices. A spanning tree must have all the vertices (that 24 of them), and as it is a tree, it must have **23** *edges.*

[9] (b) Apply Kruskal's algorithm to find a minimal spanning tree.

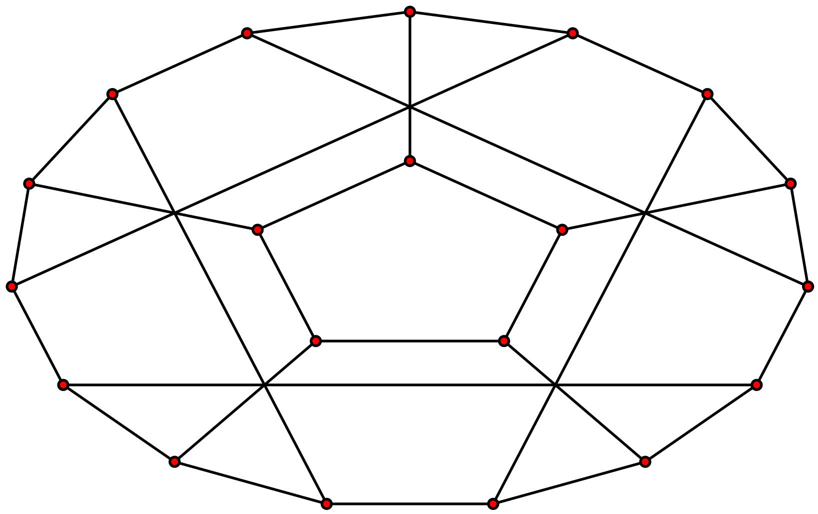
- Emphasize edges of the tree, and label them e_1, e_2, ..., e_k in the order you add them to the tree (k is the number you found in part (a)).
- Write the weight of the tree found.
- Explain in detail how did you choose e_1 and e_k.

The weight of the tree is $4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3 + 4 \cdot 4 + 4 \cdot 5 + 6 + 7 + 8 = 81$. (There are many possibilities for such graph.)

We have chosen e_1 as any of the edges of minimum weighth, that is weighth 1.

When we were choosing e_{23}, all edges of weight < 8 would have created a cycle, so we needed to choose any of the four edges of weighth 8.

[10] **Bonus.** Decide whether the graph on the figure contains a Hamilton cycle.
Warning 1: It is not easy.
Warning 2: Partial credit will be assigned only for substantial part of the solution.
If you have other questions to work upon, they are likely to bring you more points.



The graph contains no Hamilton cycle. (It contains easy to find Hamilton paths though, but no points for that :–)).

For proving it there is no really easy method. A straightforward (but a bit tedious) way is to use the method we saw in class (and which is easy to rediscover ...) — you start somehow, at some point you try all possibilities, and you keep track of all the "forcing" that your choices implied. It is also possible to use the rotational symmetry of the graph to make it a bit easier. Still, even if you know well what you are doing and make optimal decisions about which places to make choices at, it takes at least four completely diffent pictures (and at lot of thinking). You have been warned.

A fancy way of solving this problem involves realizing, that if a 3-regular graph contains a Hamilton cycle, then it is actually possible to color the edges by 3 colors, so that at each vertex meet 3 edges of distinct colors. This part is simple: just use alternately two colors along the cycle, then use the last color for the rest. Now you can be trying to find this 3-coloring of edges, which seems to be a bit easier to work with. After some case checking, it is possible to see that the given graph does not have such coloring.

The details are omitted.