SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Final Exam

MACM 201 Summer 2007

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August 7, 2007, 8:30 – 11:30

1. Suppose we want to distribute 10 (distinct) books among 4 students.

[5] (a) How many ways are there to do this if we want everyone to get at least one book?

Let S be the set of all possible assignments of books to people— we can represent this as functions, that is

$$S = \big\{ f : \{1, 2, \dots, 10\} \to \{1, 2, 3, 4\} \big\} .$$

For each $i = 1, \dots, 4$ we let c_i be property of elements of S: "the i-th person gets no book", that is

$$f \text{ satisfies } c_i \text{ iff for every } x = 1, \dots, 10 \text{ we have } f(x) \neq i .$$

We will use the Principle of Inclusion and Exclusion to find the number of elements of S that satisfy at least one of the conditions c_i, that is we want to find $N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ \ \overline{c_4}\ )$.

First, $S_0 = |S| = 4^{10}$. Further, for any $k = 1, \dots, 4$, $N(c_1 c_2 \dots c_k) = (4-k)^{10}$, moreover the same is true for any choice of k conditions among $c_1, \dots, c_4$. Thus we have $S_k = \binom{4}{k}(4-k)^{10}$. By the Principle of Inclusion and Exclusion we find the answer

$$S_0 - S_1 + S_2 - S_3 + S_4 = 4^{10} - \binom{4}{1}3^{10} + \binom{4}{2}2^{10} - \binom{4}{3}1^{10} + 0 .$$

(By the way, the numerical answer is $818\,520$.)

[5] (b) How many ways are there to do this if we want exactly two people to get nothing? At least two people to get nothing?

If we want exactly two people to get nothing, we have

$$E_2 = S_2 - 3S_3 + \binom{4}{2}S_4 = \binom{4}{2}2^{10} - 3\binom{4}{3}1^{10}(= 6132)$$

possibilities. If we want at least two people to get nothing, the number of possibilities is

$$L_2 = S_2 - 2S_3 + 3S_4 = \binom{4}{2}2^{10} - 2\binom{4}{3}1^{10}(= 6136)$$

[10] **2.** A deck of cards contains 52 cards—4 of each of the 13 ranks. What is the number of ways to order the cards in a line, so that no 4 cards of the same rank appear as a consecutive group? (For example, we don't want 10 J 9 9 9 9 K A ...)

Let S consist of all permutations of the cards. Let for $i = 1, \ldots, 13$ the condition c_i be that the permutation contains the four cards of type i as a consecutive group. (We can number the types arbitrarily, for example type 1 is A, type 2 is 2, ..., type 13 is K.)

We will use the Principle of Inclusion and Exclusion to find the number of elements of S that satisfy none of the conditions c_i. For this, we will need the following:

- $S_0 = N = |S| = 52!$ (the cards of the same rank are distinguished by the suit)
- For $k = 1, \ldots, 13$, $N(c_1 \ldots c_k) = (52 - 4k + k)!4!^k$: we treat the k four-tuples of cards (that should be kept together) as a single object, so we are permuting $52 - 4k$ cards plus k groups, each of which can be further permuted.
- Due to symmetry, $S_k = \binom{13}{k} N(c_1 \ldots c_k) = \binom{13}{k}(52 - 4k + k)!4!^k$.

Consequently, we obtain the answer

$$N(\ \overline{c_1}\ \ \overline{c_2}\ \ldots\ \overline{c_{13}}\ ) = \sum_{k=0}^{13} (-1)^k S_k = \sum_{k=0}^{13} (-1)^k \binom{13}{k} (52 - 4k + k)!4!^k$$

(About 99.77% of all permutations.)

[10] **3.** Solve the following recurrence relation

$$a_{n+2} = a_{n+1} + 2a_n + 2n + 1 \quad (n \geq 0), \qquad a_0 = 0, a_1 = 1\,.$$

The characteristic equation is $t^2 - t - 2 = 0$, its roots are $t_1 = 2$, $t_2 = -1$. So, the general homogeneous solution is of form $a_n^{(h)} = A2^n + B(-1)^n$. We will look for a particular solution in the form $a_n^{(p)} = Cn + D$ (C, D suitable constants). Plugging in the recurrence relation we obtain

$$C(n+2) + D - (C(n+1) + D) - 2(Cn + D) = 2n + 1$$

A simple (in fact, the only) way how to satify this equation for every n is solving a system of two equations

$$\begin{aligned} n(C - C - 2C) &= 2n \\ 2C + D - C - D - 2D &= 1 \end{aligned}$$

This gives us $C = D = -1$. We've found $a_n^{(p)} = -n - 1$. So the general solution is $a_n = a_n^{(h)} + a_n^{(p)} = A2^n + B(-1)^n - n - 1$. To determine A and B we put $n = 0$ and $n = 1$:

$$\begin{aligned} 0 &= a_0 = A + B - 1 \\ 1 &= a_1 = 2A - B - 2 \end{aligned}$$

this gives $A = 4/3$ and $B = -1/3$.

Altogether, we found that

$$a_n = \frac{4}{3}2^n - \frac{1}{3}(-1)^n - n - 1 \qquad (n \geq 0).$$

4. Find generating functions for the following sequences (include all the necessary computation).

[1] (a) $1, 2, 4, 8, 16, \ldots$

$1 + 2x^1 + 4x^2 + 8x^3 + 16x^4 + \cdots = \frac{1}{1-2x}$ (geometric series with quotient $2x$ and initial term 1)

[1] (b) $1+1, 1+a, 1+a^2, 1+a^3, 1+a^4, \ldots$ (a is a parameter)

$$\begin{aligned}(1 &+ 1) + (1+a)x^1 + (1+a^2)x^2 + (1+a^3)x^3 + (1+a^4)x^4 + \cdots \\ &= 1 + 1 + 1x^2 + 1x^3 + 1x^4 + \cdots \\ &\quad + 1 + ax^1 + a^2x^2 + a^3x^3 + a^4x^4 + \cdots \\ &= \frac{1}{1-x} + \frac{1}{1-ax}\end{aligned}$$

(sum of two geometric series)

[2] (c) $0, 1, 2, 3, 4, 5, 6, 7, \ldots$

$\frac{x}{(1-x)^2}$ (we had this in class, or you may derive it again, e.g. by considering the derivative of $\frac{1}{1-x}$).

[3] (d) Find the coefficient of x^{14} in $(x^2 + \frac{2}{x})^{40}$.

The k-th term in binomial theorem applied to our expression is $\binom{40}{k} x^{2k} (2/x)^{40-k}$. We need such k that $2k - (40-k) = 14$, which is $k = 54/3 = 18$. Thus the coeffiecient is $\binom{40}{18} 2^{22}$.

[3] (e) Let a_n be the number of ways to write n as a sum of distinct odd positive integers (disregarding the order). What is the generating function for the sequence $a_0, a_1, a_2, \ldots$? (Explain what various parts of the function correspond to.)

$$(1+x)(1+x^3)(1+x^5)(1+x^7)(1+x^9)\cdots.$$

The first part corresponds to the choice of having zero or one times a 1 in our sum, the second part corresponds to the number of copies of 3, the third to the 5's etc.

[10] **5.** We need to distribute 120 envelopes among 4 students so that everybody gets at least 6 and at most 40 envelopes. How many ways are there to do this?

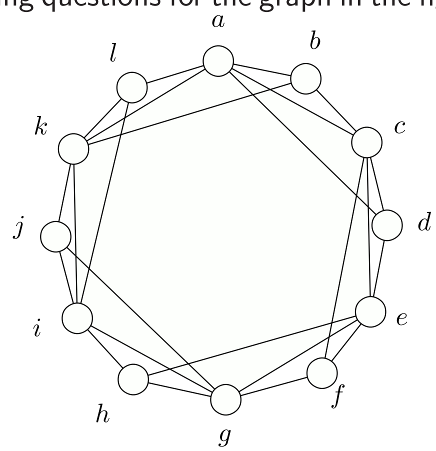
Solution 1: The required number can be expressed as

$$\begin{aligned}
& [x^{120}](x^6+\cdots+x^{40})^4 \\
= \;& [x^{120}]\left(\frac{x^6-x^{41}}{1-x}\right)^4 \\
= \;& [x^{120}]x^{24}\left(\frac{1-x^{35}}{1-x}\right)^4 \\
= \;& [x^{96}]\frac{1-4x^{35}+\binom{4}{2}x^{70}-\binom{4}{3}x^{105}+\binom{4}{4}x^{140}}{(1-x)^4} \\
= \;& \binom{96+3}{3}-4\binom{96-35+3}{3}+\binom{4}{2}\binom{96-2\cdot 35+3}{3} \\
= \;& \binom{99}{3}-4\binom{64}{3}+\binom{4}{2}\binom{29}{3} \\
= \;& 12\,117\,.
\end{aligned}$$

We have used standard operations with generating functions and the binomial theorem in the following form: $[x^n](1-x)^{-k}=\binom{n+k-1}{k-1}$. The numerical answer was not needed in the exam.

Solution 2: Using the PIE. We are looking for solutions to $s_1+\cdots+s_4=120$, where $6\le s_i\le 40$. By substitution, we may look for solution to $s_1+\cdots+s_4=96$, where $0\le s_i\le 34$. If we let S to be set of all such solutions and c_i the condition that $s_i\ge 35$, we are set to apply the principle, which gives us the same formula as above.

6. Answer the following questions for the graph in the figure. (**No explanation needed.**)



[1] (a) How many vertices does the graph have? **12**

[1] (b) How many edges does the graph have? **24**

[1] (c) If the graph was drawn without crossings, how many faces would it have?
$\mathbf{f = e - v + 2 = 14}$

[2] (d) How many *induced* subgraphs contain the edge ab?
$\mathbf{2^{10}}$ (as each such subgraph corresponds to a subset containing a, b and any number of the other 10 vertices).

[2] (e) How many *spanning* subgraphs contain the edge ab?
$\mathbf{2^{23}}$ (as each such subgraph corresponds to a subset of the edge-set of the graph: it must contain ab and may contain any of the other 23 edges).

[1] (f) Specify a subgraph that is not induced.
For example (V, E) where $V = \{a, b, c\}$, $E = \{ab, bc\}$. (Many possibilities exist.)

[1] (g) How many edges does a spanning tree have? $\mathbf{11 (= 12 - 1)}$

[1] (h) What is the distance (length of a shortest path) between a and g? **3**

7. Decide, what is the maximal length of a

1. cycle
2. circuit

in the following graphs. Explain your results!

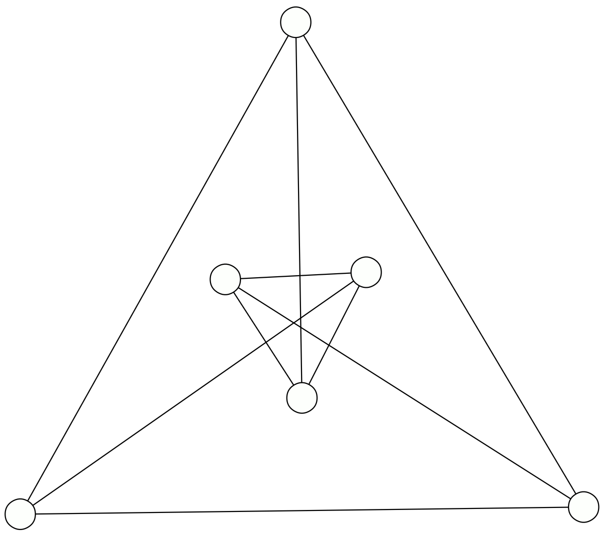
[5] (a) K_9

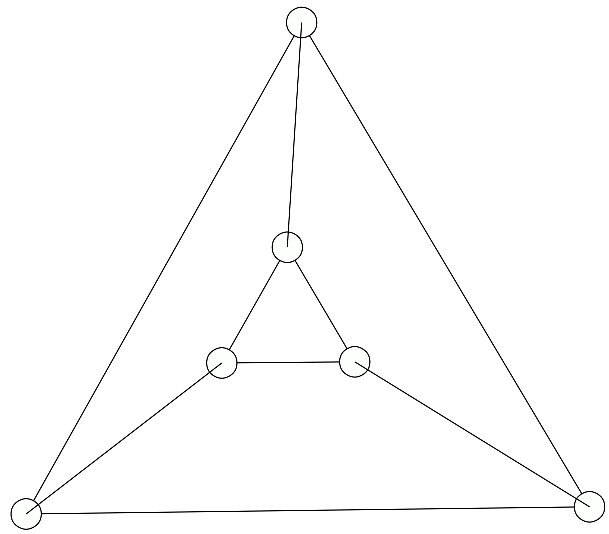
1. The graph contains a Hamilton cycle (choose any 9 vertices, connect them in any order), so the maximal length is 9.
2. The graph contains an Euler circuit, as it is connected and every vertex has degree 8 (even number). Thus, the maximal length is $\binom{9}{2}$ (the number of all edges of K_9).

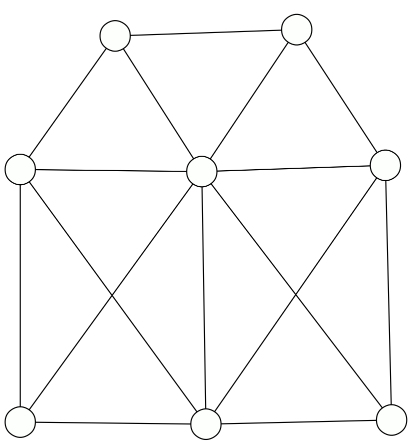
[5] (b) K_{10}

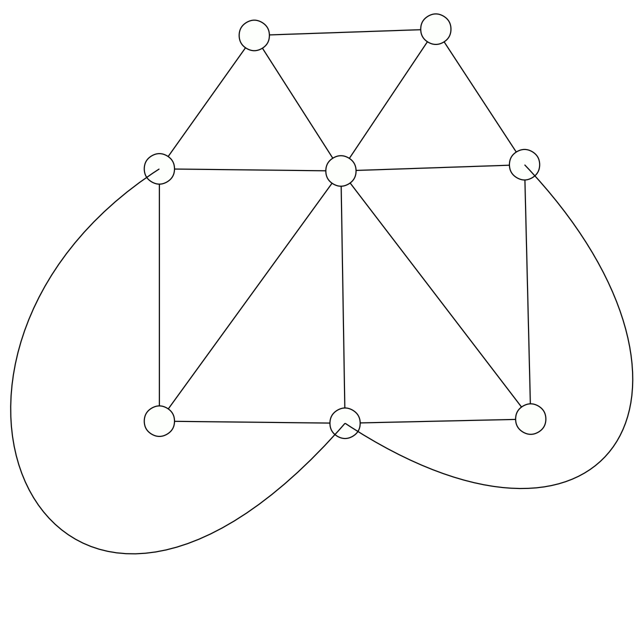
1. The graph contains a Hamilton cycle (choose any 10 vertices, connect them in any order), so the maximal length is 10.
2. Now the graph does not contain an Euler circuit: it is connected but every vertex has degree 9 (odd number). To obtain as large circuit as possible, we need to delete as few edges as possible to change the degree of every vertex to an even number (and keep the graph connected). We obviously need at least $10/2 = 5$ edges (each edge changes at most two vertices) and 5 edges also suffice: we just have to make sure to delete 5 edges such that no two share a vertex. Thus, the maximal length is $\binom{10}{2} - 5$.

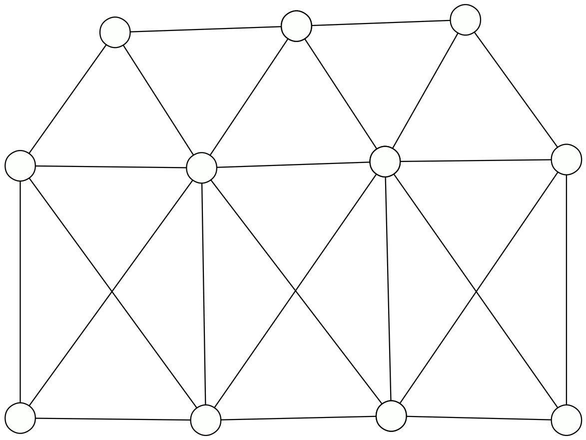
[10] **8.** Are the following graphs planar? If yes, draw the graph without crossings, if not, explain why not. [2 marks for each graph]

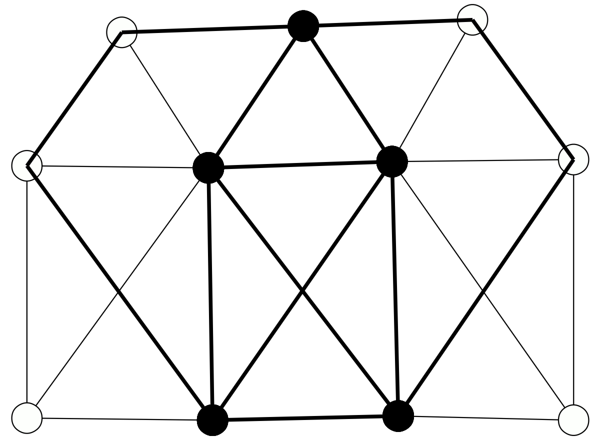


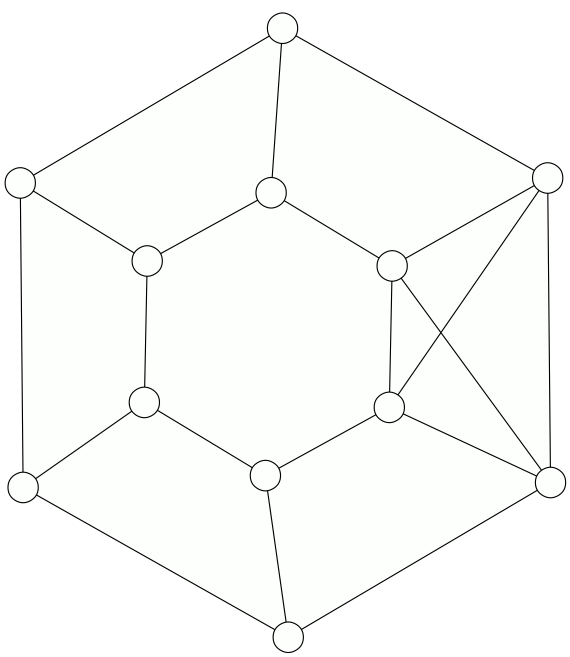


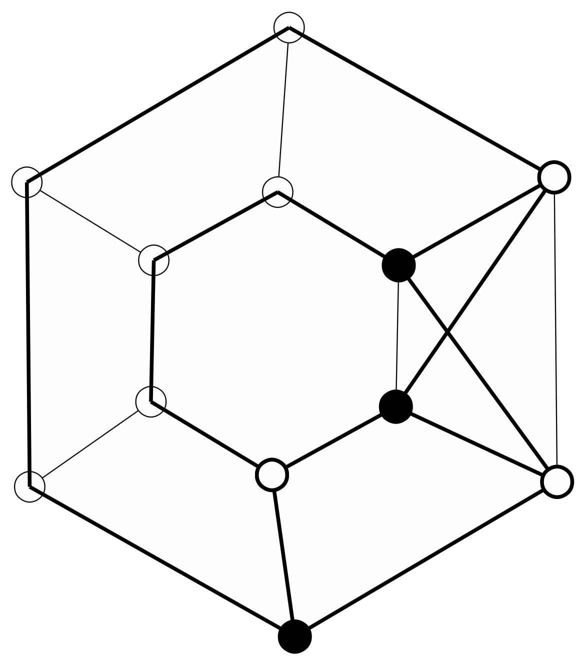


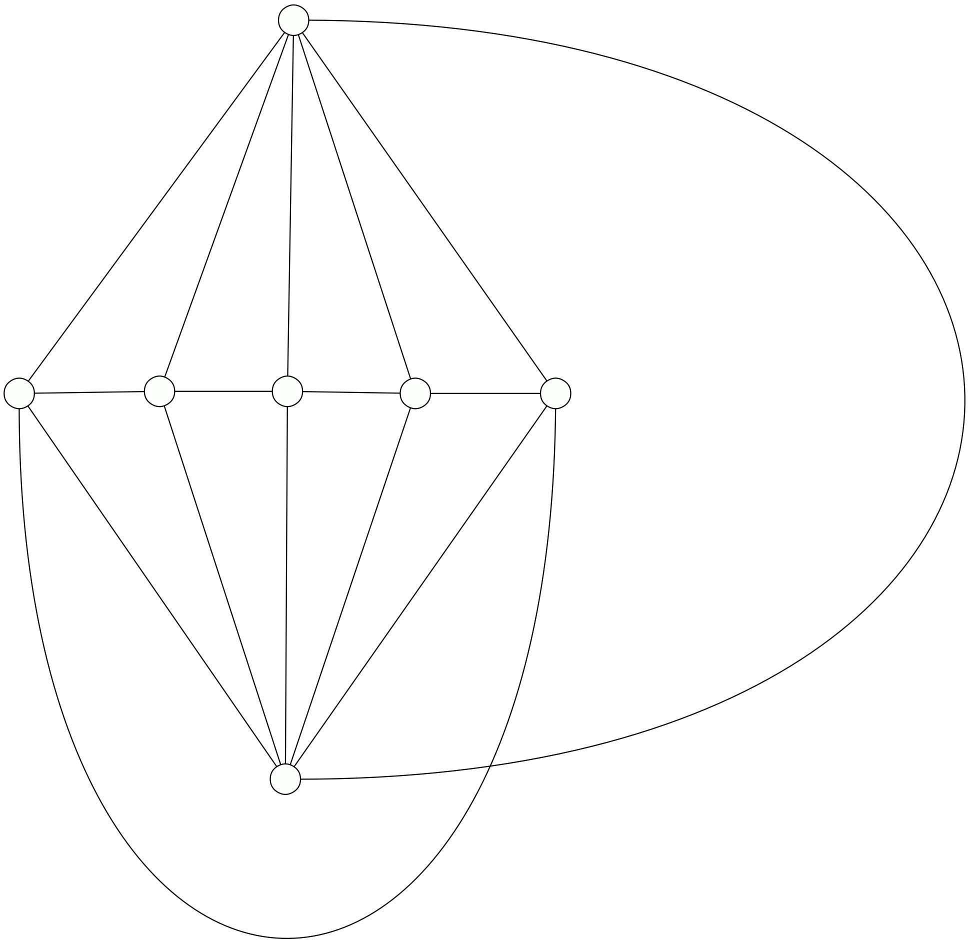






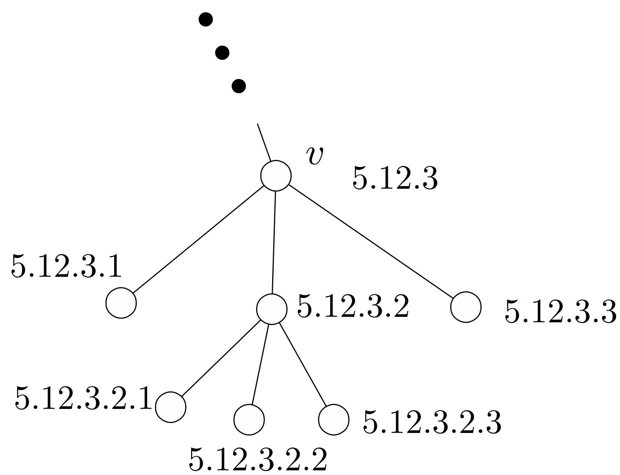






This graph has $v = 7$ vertices and $e = 3 \cdot 5 + 1 = 16$ edges. However, in a planar graph we have $e < 3 \cdot v - 6$, or $16 < 15$, which is not true—so the graph is not planar.

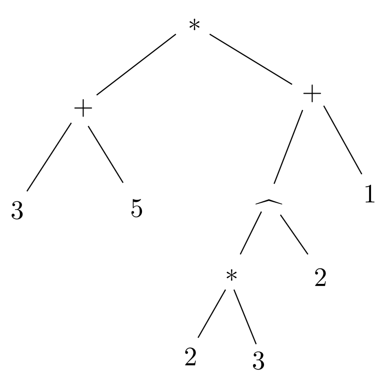
[4] **9.** (a) In the figure is *a part of* a rooted tree. (However, for each vertex in the figure all of its children are shown.) You are given the label of one of the vertices according to the universal address system. Determine the labels of the other vertices, and the labels of the ancestors of the vertex v (these ancestors are not shown on the figure).



ancestors of v: 0 (the root), 5, 5.12

[2] (b) Construct the binary rooted tree for the following arithmetic expression:

$$(3+5)*((2*3)\,\hat{}\,2+1)$$



[2] (c) Use the postorder traversal to write the expression in postfix notation.

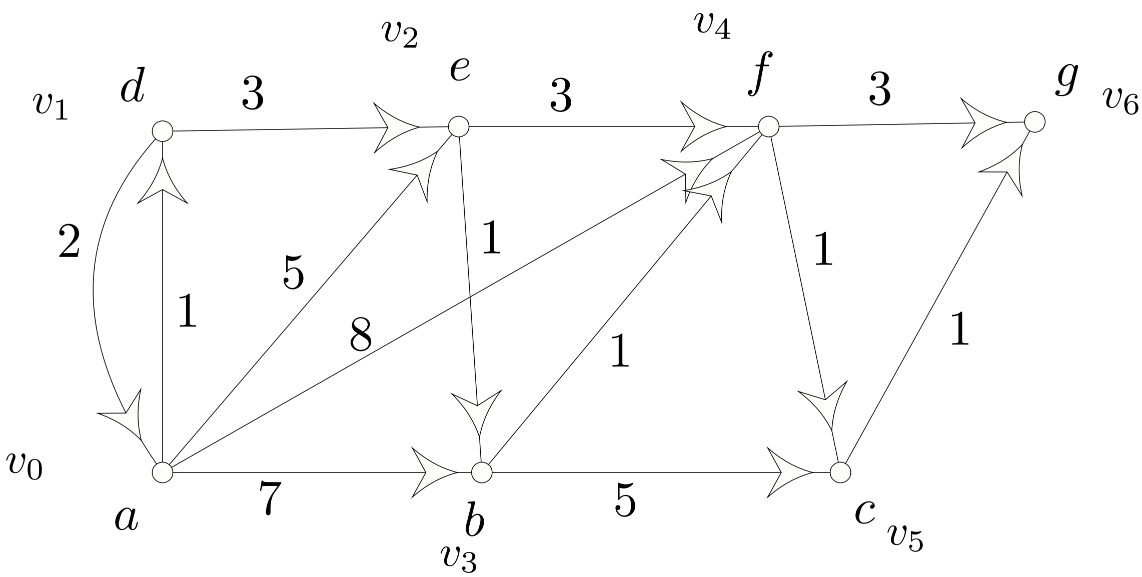
$3\ 5\ +2\ 3\ *2\,\hat{}\,1+*$

[2] (d) Use the preorder traversal to write the expression in prefix notation.

$*+3\ 5+\hat{}*2\ 3\ 2\ 1$

[10] **10.** Apply Dijkstra's algorithm to the following undirected graph in order to find the shortest path from a to each of the other vertices.

- Fill each of the determined distances in the table below and provide a shortest path from a to g.
- Label the vertices v_0, ..., v_6 according to the algorithm (in the order in which you reached the "final" decision about the distance of the vertex from a).
- Explain in detail the first three steps of the algorithm (that is, all considerations before v_3 was determined).



a	b	c	d	e	f	g
0	5	7	1	4	6	8

The shortest a-g path is: a-d-e-b-f-c-g.

Beginning of the algorithm:

$L(v) \setminus v$	a	b	c	d	e	f	g	
start	$(0, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	
1. step	$(0, -)$	$(7, a)$	$(\infty, -)$	$(1, a)$	$(5, a)$	$(8, a)$	$(\infty, -)$	$\Rightarrow v_1 = d$
2. step	$(0, -)$	$(7, a)$	$(\infty, -)$	$(1, a)$	$(4, d)$	$(8, a)$	$(\infty, -)$	$\Rightarrow v_2 = e$
3. step	$(0, -)$	$(5, e)$	$(\infty, -)$	$(1, a)$	$(4, d)$	$(7, e)$	$(\infty, -)$	$\Rightarrow v_3 = b$

Bonus. Let $m, n \geq 1$ be integers. Let $G_{m,n}$ be the following graph. The vertex-set is $V = \{(x, y) \mid 1 \leq x \leq m, 1 \leq y \leq n\}$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if

- $y_1 = y_2$ or
- $x_1 = x_2$ and $|y_1 - y_2| \in \{1, n-1\}$

m and n are switched at most places!!!

[5] (a) For which integers m, n does $G_{m,n}$ have an Euler circuit? Explain!

Each vertex is connected to two vertices with the same x-coordinate and $n-1$ with the same y-coordinate. Thus, we need $n-1+2 = n+1$ to be even, or n to be odd. (The graph is connected: every vertex is connected to some vertex with the x-coordinate equal to 1, and these vertices induces a cycle.) Thus, $G_{m,n}$ has an Euler circuit if and only if n is odd.

[5] (b) For which integers m, n does $G_{m,n}$ have a Hamilton cycle? Explain!

The idea is (similarly to the exercise showing the hypercube to be hamiltonian), to connect several "parts of a hamilton cycle" in various "levels" of the graph. The vertices with a fixed y-coordinate induce a complete graph K_m, so any ordering of the vertices forms a paths; this will come in handy.

For $n = 1$, the graph $G_{m,1}$ is isomorphic to K_m, so it has a Hamilton cycle iff $m \geq 3$. If n is bigger, we will use sort of "recursive construction", again assuming $m \geq 3$:

1. we choose a path from $(1, 1)$ to $(2, 1)$ of length $m-1$, using only vertices with y-coordinate equal to 1 (this will use all of them)
2. the next vertex will be $(2, 2)$
3. we choose a path from $(2, 2)$ to $(3, 2)$ of length $m-1$, using only vertices with y-coordinate equal to 2 (this will use all of them)
4. the next vertex will be $(3, 3)$
5. next we choose a path from $(3, 3)$ to $(2, 3)$, etc.
6. finally, we find a path from $(2, n)$ (or $(3, n)$, depending on parity of n) to $(1, n)$, and finish the cycle to $(1, 1)$.

(You should check, that all edges that we need are actually in the graph.) This works for any $n \geq 2$ and $m \geq 3$.

If $m = 1$, the graph $G_{1,n}$ is isomorphic to C_n, so it has a Hamilton cycle iff $n \geq 3$. For $m = 2$ we have two copies of C_n (one consisting of vertices with x-coordinate equal to 1, the other to 2). We delete "the same" edge from both copies of C_n and connect those copies together. This works for any $n \geq 2$.

To sum it up, $G_{m,n}$ contains a Hamilton cycle in all cases except $(m, n) = (1, 1), (1, 2), (2, 1)$.