SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Final Exam – Solutions

MACM 201 Spring 2007

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April 11, 2007, 8:30 – 11:30

[10] **1.** Suppose everyone can choose one of the 12 calendar months to take their holidays. If each person in a group of 30 does this, how many ways will there be so that there is at least 1 month when none of them will be taking holidays?

Let S be the set of all possible assignments of months to people— we can represent this as functions, that is

$$S = \left\{ f : \{1, 2, \dots, 30\} \to \{1, 2, \dots, 12\} \right\}.$$

For each $i = 1, \dots, 12$ we let c_i be property of elements of S: "the i-th months is not assigned to anyone", that is

$$f \text{ satisfies } c_i \text{ iff for every } x = 1, \dots, 30 \text{ we have } f(x) \neq i\,.$$

We will use the Principle of Inclusion and Exclusion to find the number of elements of S that satisfy at least one of the conditions c_i, that is we want to find $|S| - N(\ \overline{c_1}\ \ \overline{c_2}\ \dots\ \overline{c_{12}}\ )$.

First, $N = |S| = 12^{30}$. Further, for any $k = 1, 2, \dots, 12$, $N(c_1 c_2 \dots c_k) = (12-k)^{30}$, moreover the same is true for any choice of k conditions among c_1, …, c_{12}. Thus we have $S_k = \binom{12}{k}(12-k)^{30}$. By the Principle of Inclusion and Exclusion we find the answer

$$S_1 - S_2 + S_3 - \cdots + S_{11} - S_{12} = \sum_{k=1}^{12} \binom{12}{k} (12-k)^{30} (-1)^{k+1}\,.$$

This is, by the way, about 64% of all 12^{30} possibilities (however this numerical result was not asked for).

2. We say a word W contains a word W' if W' is included in W without gaps: e.g., the word KAYAK contains YAK but not KYK. Consider all words obtained by reordering the letters of the word ADVENTURE. How many of these words contain

[5] (a) none of the words VAN, AD, TEN, RED?

Let S consist of all permutations of the letters of the word ADVENTURE, let a word W satisfy

- c_1 iff W contains VAN,
- c_2 iff W contains AD,
- c_3 iff W contains TEN, and
- c_4 iff W contains RED.

We will use the Principle of Inclusion and Exclusion to find the number of elements of S that satisfy none of the conditions c_i. For this, we will need

- $S_0 = N = |S| = 9!/2$ (permutations with repetitions)
- To find all words containing VAN, we permute one 'object' VAN, 2 letters E and 4 more (pairwise distinct) letters. So, $N(c_1) = (1+2+4)!/2 = 7!/2$
- By similar argument, $N(c_2) = 8!/2$, and $N(c_3) = N(c_4) = 7!$.
- Thus, $S_1 = N(c_1) + N(c_2) + N(c_3) + N(c_4) = 7!/2 + 8!/2 + 2 \cdot 7!$.
- No word contains both VAN and AD (there are not enough A's). Hence $N(c_1c_2) = 0$, similarly $N(c_1c_3) = N(c_2c_4) = 0$. For counting $N(c_1c_4)$ we permute two 'objects' VAN and RED, and 3 more letters (T, U, and E), thus $N(c_1c_4) = 5!$. Similarly, $N(c_2c_3) = 6!$, and $N(c_3c_4) = 5!$.
- So, $S_2 = N(c_1c_2)+N(c_1c_3)+N(c_1c_4)+N(c_2c_3)+N(c_2c_4)+N(c_3c_4) = 0+0+5!+6!+0+5!$.
- No word satisfies more then two of the conditions, hence $S_3 = S_4 = 0$ (as $N(c_1c_2) = N(c_1c_3) = N(c_2c_4) = 0$).

Consequently, we obtain the answer

$$N(\ \overline{c_1}\ \ \overline{c_2}\ \ \overline{c_3}\ \ \overline{c_4}\ ) = S_0 - S_1 + S_2 - S_3 + S_4 = 9!/2 - (7!/2 + 8!/2 + 2 \cdot 7!) + (2 \cdot 5! + 6!)\,.$$

(The numerical value is 149640: about 82% of all $9!/2$ possibilities.)

[5] (b) at least two of those words?

By the generalized version of the Principle of Inclusion and Exclusion, this is $L_2 = S_2 - 2S_3 + 3S_4$, but as $S_3 = S_4 = 0$, we have just $L_2 = S_2 = 2 \cdot 5! + 6!$.

(Which is 960.)

3. Fifty different books (10 on each of five topics) are placed on five bookshelves (each topic on one of the shelves). All books are removed from the shelves and then put back, again all books on the same topic are placed on one shelf. How many ways are there to choose the new positions of the books, so that

[4] (a) no topic is put on the same shelf as before?

Recall that the number of derangements of n objects (that is, permutations with no fixed point) is $d_n = n!(1 - 1/1! + 1/2! - 1/3! + \cdots + (-1)^n/n!)$.

We have d_5 ways how to choose the new shelf for each of the topics, and $10!$ possible permutations of the books of each of the five topics. Hence the answer is

$$d_5 \cdot (10!)^5 = 5! \cdot (1/2 - 1/6 + 1/24 - 1/120) \cdot (10!)^5$$

(by the way, $d_5 = 44$ and $10! \doteq 3.6{\cdot}10^6$, the final answer is 27686486176707494459473920000000000).

[6] (b) no topic is put to the same shelf as before **and** no book is placed to the same position within a shelf, as before? (For example, if a book was at the second place from the left on a shelf, then it may not be placed to second place on any shelf.)

Now we have d_5 ways to choose the shelves and for each of the five topics d_{10} ways to arrange the books of that topic. Hence we have

$$d_5 \cdot (d_{10})^5 = 5! \cdot (1/2 - 1/6 + 1/24 - 1/120) \cdot \big(10! \cdot (1/2 - 1/6 + \cdots + 1/10!)\big)^5$$

(btw, this is about $e^{-5} = 0.6\%$ of the answer to part a).

[10] **4.** How many ways are there to pay 20 dollars in 25-cent, 1-dollar, and 2-dollar coins? Use generating functions! (You don't need to find a numerical answer, a simple formula involving factorials, binomial coefficients, etc. is fine.)

Let us denote the number of quarters, loonies, and toonies by a, b, and c. We are looking for a, b, and c such that $a/4 + b + 2c = 20$, or

$$a + 4b + 8c = 80\,.$$

This means, we are interested in the following quantity

$$\begin{aligned}
& [x^{80}](1 + x + \cdots)(1 + x^4 + \cdots)(1 + x^8 + \cdots) \\
= \ & [x^{80}]\frac{1}{1-x}\cdot\frac{1}{1-x^4}\cdot\frac{1}{1-x^8} \\
= \ & [x^{80}]\frac{(1 + x + x^2 + \cdots + x^7)(1 + x^4)}{(1-x^8)^3} \\
= \ & [x^{80}]\frac{1 + \cdots + x^8 + \cdots + x^{11}}{(1-x^8)^3} \\
= \ & [x^{80}]\frac{1}{(1-x^8)^3} + [x^{72}]\frac{1}{(1-x^8)^3} \\
= \ & \binom{-3}{10} + \binom{-3}{9} = \binom{12}{10} + \binom{11}{9} \\
= \ & \binom{12}{2} + \binom{11}{2} = 121
\end{aligned}$$

(We have used the binomial theorem and standard methods to modify a generating function. The numerical answer was optional.)

[10] **5.** Solve the following recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = 2n + 1 \quad (n \geq 0), \qquad a_0 = 0, a_1 = 0\,.$$

The characteristic equation is $t^2 - 5t + 6 = 0$, its roots are $t_1 = 2$, $t_2 = 3$. So, the general homogeneous solution is of form $a_n^{(h)} = A2^n + B3^n$. We will look for a particular solution in the form $a_n^{(p)} = Cn + D$ (C, D suitable constants). Plugging in the recurrence relation we obtain

$$C(n+2) + D - 5(C(n+1) + D) + 6(Cn + D) = 2n + 1$$

A simple (in fact, the only) way how to satify this equation for every n is solving a system of two equations

$$\begin{aligned} n(C - 5C + 6C) &= 2n \\ 2C + D - 5C - 5D + 6D &= 1 \end{aligned}$$

This gives us $C = 1$, $D = 2$. We've found $a_n^{(p)} = n + 2$. So the general solution is $a_n = a_n^{(h)} + a_n^{(p)} = A2^n + B3^n + n + 2$. To determine A and B we put $n = 0$ and $n = 1$:

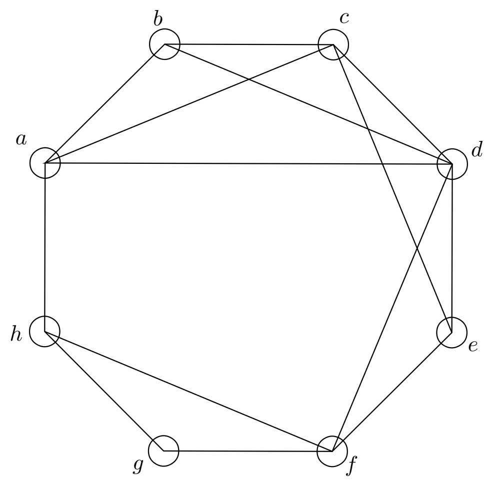
$$\begin{aligned} 0 &= a_0 = A + B + 2 \\ 0 &= a_1 = 2A + 3B + 3 \end{aligned}$$

this gives $A = -3$ and $B = 1$.

Altogether, we found that

$$a_n = -3 \cdot 2^n + 3^n + n + 2 \qquad (n \geq 0).$$

6. Answer the following questions for the graph on the figure. (No explanation needed.)



[1] (a) How many loops does the graph have? **0**

[1] (b) Does the graph have multiple edges? **No.**

[2] (c) How many 3-cycles does the graph have? (3-cycles having the same vertices and edges count only once).

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[2] (d) How many induced subgraphs does the graph have?

$2^8 - 1$ **(or** 2^8**, if you do count the graph with zero vertices).**

[1] (e) Specify a subgraph that is not induced.

$V = \{a, b, c\}$, $E = \{ab, bc\}$

[1] (f) Write down edges of a spanning tree.

$ab, bc, cd, de, ef, fg, gh$

[1] (g) Write down a path of length 3 (that is, with three edges) between vertices a and b.

a-d-c-b **or** a-c-d-b

[1] (h) How many edges does the complement of the graph have? **14**

[5] **7.** (a) Recall that for $n \geq 1$ the n-dimensional hypercube is the graph $Q_n = (V, E)$, where $V = \{0,1\}^n$, and for $x, y \in V$, we have $\{x, y\} \in E$ if and only if x and y differ in exactly one coordinate.
For which integers n does Q_n have an Euler circuit? Explain!

- Every vertex of Q_n has degree n: A vertex $v = (v_1, v_2, \ldots, v_n)$ is connected to the following vertices: $v^i = (v_1, v_2, \ldots, v_{i-1}, 1 - v_i, v_{i+1}, \ldots, v_n)$ (for every $i = 1, 2, \ldots, n$).
- Each of the graphs Q_n is connected: every vertex is connected to $(0, 0, \ldots, 0)$ by a path of length at most n (we change the nonzero coordinates to 0, one at a time).

By a theorem from class it follows, that Q_n has an Euler circuit, if and only if n is even.

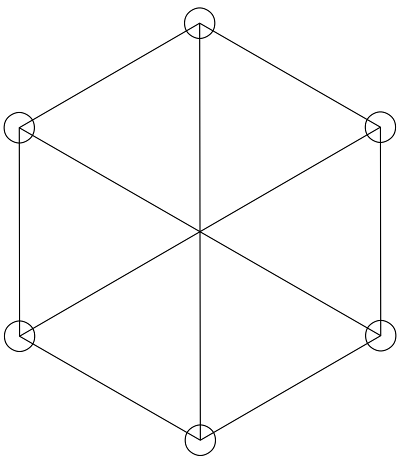
[5] (b) For which integers n does Q_n have a Hamilton cycle? Explain!

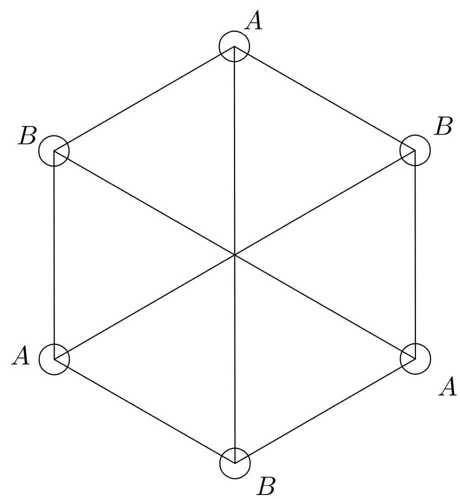
If $n = 1$, then $Q_n \simeq K_2$ does not have a Hamilton cycle.

If $n = 2$, then $Q_n \simeq C_4$ does have a Hamilton cycle.

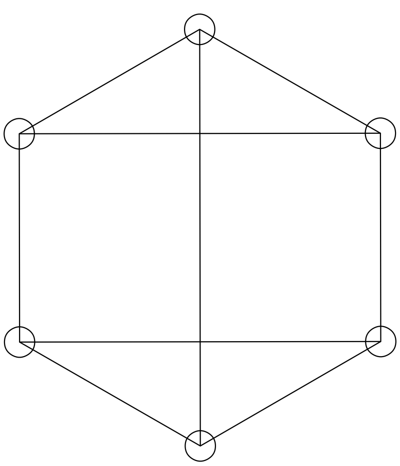
By induction we prove that the same is true for every $n \geq 2$. We already know it for $n = 2$, so we only need to show that Q_{n+1} has a Hamilton cycle, assuming that Q_n does. To do so, pick any Hamilton cycle $v_1, v_2, \ldots, v_{2^n}$ in Q_n. Now recall that Q_{n+1} consists of two copies of Q_n plus some edges between. In particular, all edges $\{v_i 0, v_{i+1} 0\}$ and $\{v_i 1, v_{i+1} 1\}$ are present in Q_{n+1} (these edges form copies of the chosen cycle, with one edge omitted). After adding edges $\{v_1 0, v_1 1\}$ and $\{v_{2^n} 0, v_{2^n} 1\}$ we obtain a Hamilton cycle in Q_{n+1}.

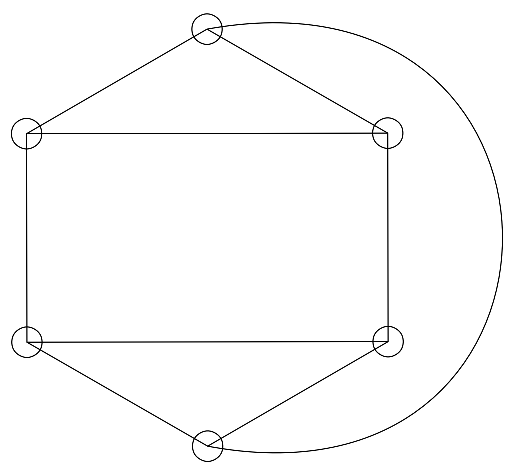
[10] **8.** Are the following graphs planar? If yes, draw the graph without crossing, if not, explain why not. [2 marks for each graph]



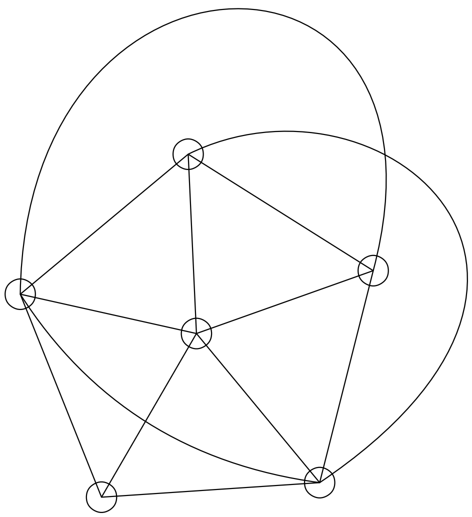


The graph is not planar: this is $K_{3,3}$, which we discussed in class.

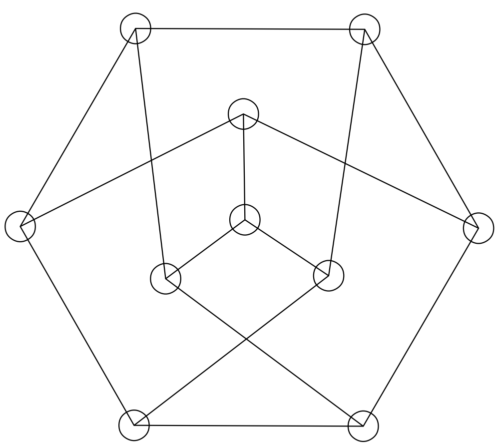


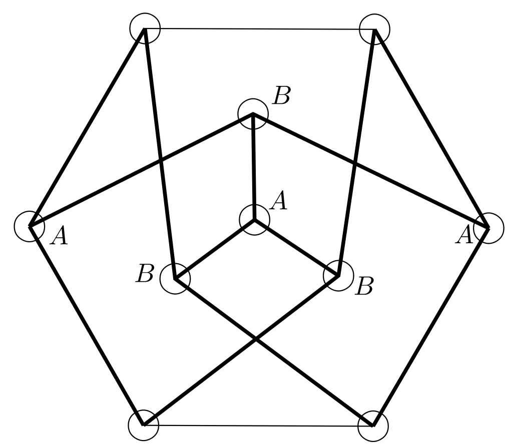


The graph is planar.

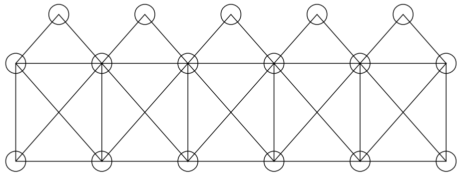


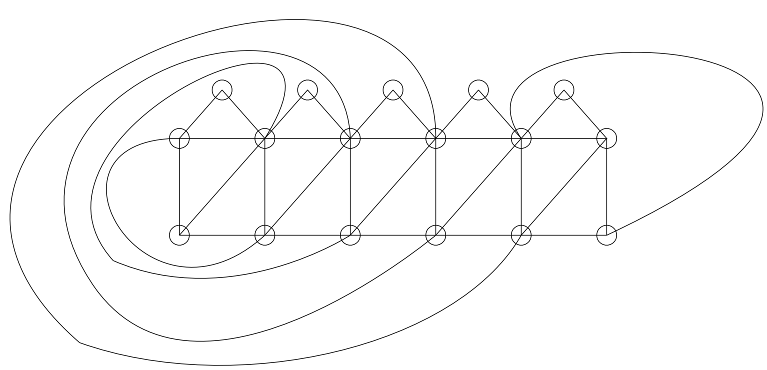
The graph is not planar: it has $v = 6$ vertices, $e = 13$ edges, and $e > 3v - 6$.





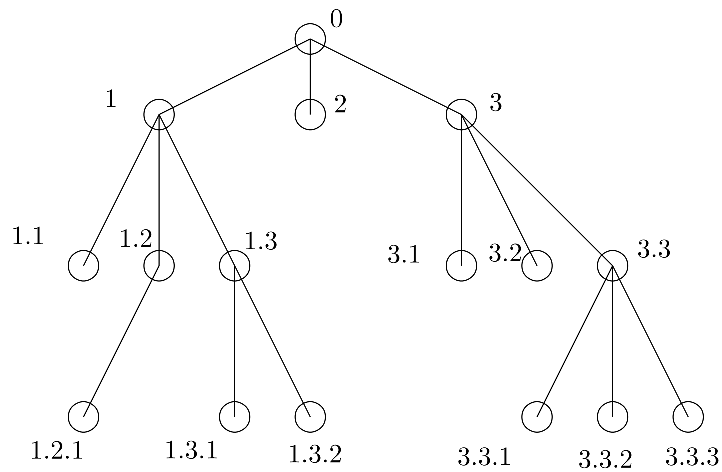
The graph is not planar: we use Kuratowski's theorem, a subgraph homeomorphic to $K_{3,3}$ is depicted.





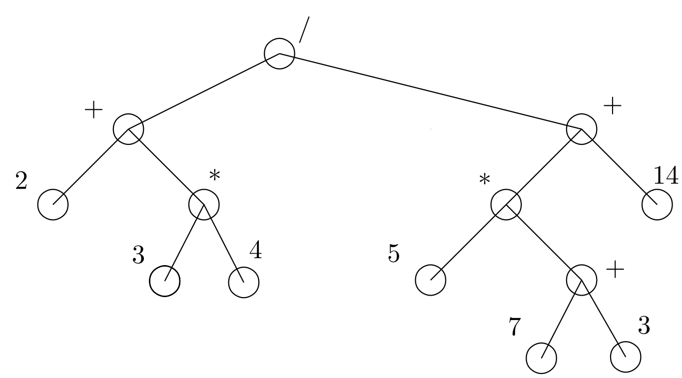
The graph is planar.

[4] **9.** (a) For the rooted tree on the figure, determine the labels of the vertices according to the universal address system.



[2] (b) Construct the binary rooted tree for the following arithmetic expression:

$$(2 + 3 * 4)/(5 * (7 + 3) + 14)$$



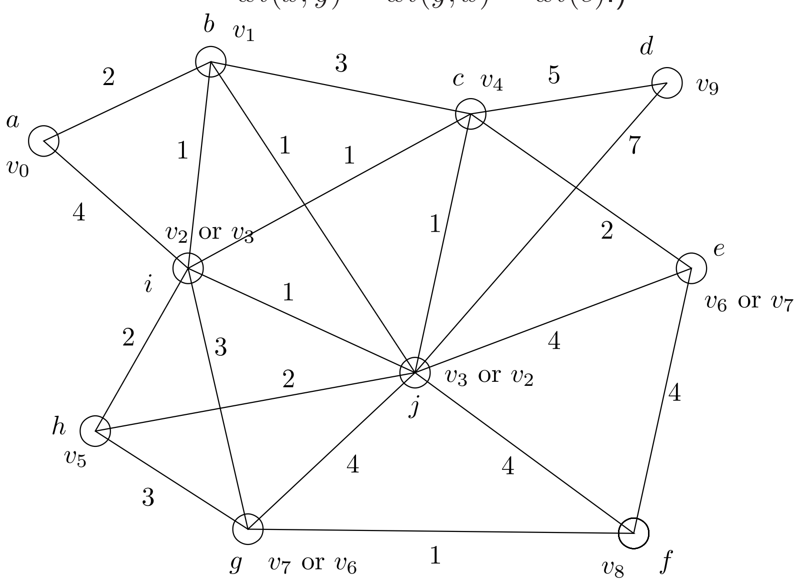
[2] (c) Use the postorder traversal to write the expression in the postfix notation.

2, 3, 4, $*$, $+$, 5, 7, 3, $+$, $*$, 14, $+$, $/$

[2] (d) Use the preorder traversal to write the expression in the prefix notation.

$/$,$+$,2,$*$,3,4,$+$,$*$,5,$+$,7,3,14

[5] **10.** (a) Apply Dijkstra's algorithm to the following undirected graph in order to find the shortest path from a to each of the other vertices. Fill each of these distances in the table below and provide a shortest path from a to f. Explain in detail the first three steps of the algorithm and label the vertices v_0, ..., v_9 according to the algorithm.
(The graph is undirected, which means that for each edge $e = \{x, y\}$ we have $wt(x, y) = wt(y, x) = wt(e)$.)



a	b	c	d	e	f	g	h	i	j
0	2	4	9	6	7	6	5	3	3

The shortest a,f-path is a-b-j-f.
Put $L(a) = (0, -)$ and $L(v) = (\infty, -)$ for all $v \neq a$. Let $F = \emptyset$.

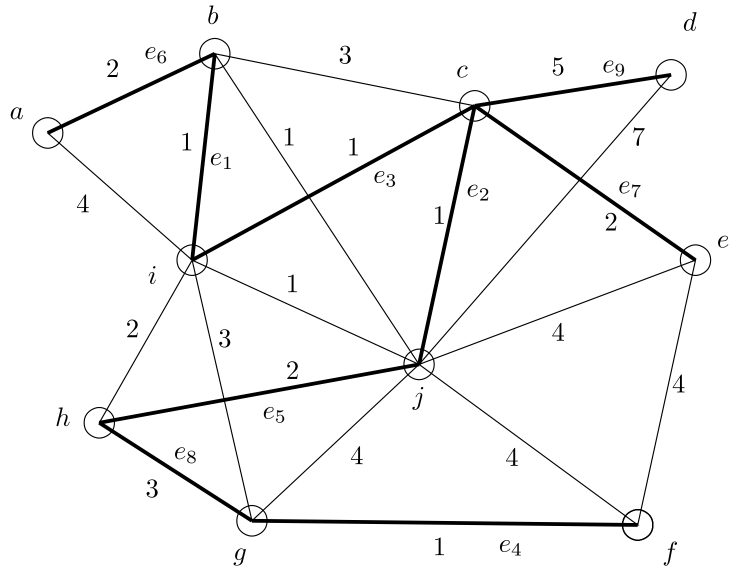
1. Pick $v_0 = a$ (minimum value of $L(v)_1$ is attained for $v = a$), $F = \{a\}$. We calculate $L(b) = (2, a)$, $L(i) = (4, a)$.

2. Pick $v_1 = b$ (minimum value of $L(v)_1$ for $v \notin F$), $F = \{a, b\}$. We calculate $L(c) = (5, b)$, $L(j) = (3, b)$, $L(i) = (3, b)$.

3. Pick $v_2 = i$ (minimum value of $L(v)_1$ for $v \notin F$), $F = \{a, b, i\}$. We calculate $L(h) = (5, i)$, $L(g) = (6, i)$, $L(c) = (4, i)$.

   We could also pick $v_3 = j$. Then $F = \{a, b, j\}$ and we calculate $L(h) = (5, j)$, $L(g) = (7, j)$, $L(f) = (7, j)$, $L(e) = (7, j)$, $L(d) = (10, j)$, $L(c) = (4, j)$.

[5] (b) Apply Kruskal's algorithm to the following graph. Determine the minimal possible sum of the weights of the edges of a spanning tree and draw an example of a tree that achieves this. Also, label the edges by e_1, ..., e_9 in the order you analyze them in the algorithm and explain in detail the first three steps of the algorithm.



1. Pick as e_1 any of the edges of the minimum weight (weight 1), e.g. $e_1 = bi$.

2. Pick as e_2 any of the remaining edges of the minimum weight, e.g. $e_2 = cj$ (and make sure we are not creating a cycle).

3. Pick as e_3 any of the remaining of the minimum weight, e.g. $e_3 = ci$.

The weight of the found tree (or of any other minimal spanning tree) is 18.

There are several optimal trees and several labelings of edges. In each of them, however edges e_1, ..., e_4 are of weight 1, e_5, e_6, e_7 of weight 2, $e_8 = hg$ or ig, $e_9 = cd$.