MACM 201 Test 2 Solutions

(1) (10) Consider the following recurrence relation;

$$a_n + 6a_{n-1} + 14a_{n-2} + 16a_{n-3} + 8a_{n-4} \;=\; 0, \quad a_0 = -1, \;\; a_1 = 1, \;\; a_2 = 2, \;\; a_3 = 3$$

Write down the general solution and the equations that will determine the unknown coefficients, <u>but do not solve for the coefficients</u>.

<u>Solution</u>: The characteristic polynomial is

$$r^4 + 6r^3 + 14r^2 + 16r + 8 \;=\; (r+2)^2(r^2+2r+2)$$

The roots are $-2, -1+i, -1-i$. We write $-1+i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$. So the general solution is

$$a_n = A(-2)^n + Bn(-2)^n + C(\sqrt{2})^n\cos\frac{3n\pi}{4} + D(\sqrt{2})^n\sin\frac{3n\pi}{4}$$

Using the initial conditions;

$$\begin{aligned} n=0 &: \;\; A + C \;=\; -1 \\ n=1 &: \;\; -2A - 2B - C + D \;=\; 1 \\ n=2 &: \;\; 4A + 8B - 2D \;=\; 2 \\ n=3 &: \;\; -8A - 24B + 2C + 2D \;=\; 3 \end{aligned}$$

(2) (10) Consider the following recurrence relation;

$$a_n - 2a_{n-1} - 3a_{n-2} \;=\; 2(3^n), \quad a_0 = 2, \quad a_1 = 3$$

(a) Use the recurrence relation to determine a_2 and a_3.

<u>Solution</u>:

$$\begin{aligned} a_2 &= 2a_1 + 3a_0 + 2(3^2) \;=\; 2(3) + 3(2) + 18 \;=\; 30 \\ a_3 &= 2a_2 + 3a_1 + 2(3^3) \;=\; 2(30) + 3(3) + 54 \;=\; 123 \end{aligned}$$

(b) Solve the recurrence relation.

<u>Solution</u>: The characteristic equation for the homogeneous relation is $r^2 - 2r - 3 = (r-3)(r+1)$, so

$$a_n^{(h)} \;=\; A(3^n) + B(-1)^n$$

The particular solution has the form

$$a_n^{(p)} \;=\; Dn(3^n)$$

We determine D by substituting $a_n^{(p)}$ into the recurrence relation;

$$\begin{aligned} Dn3^n - 2D(n-1)3^{n-1} - 3D(n-2)3^{n-2} &= 2(3^n) \\ \rightarrow \quad D &= \frac{3}{2} \end{aligned}$$

So the general solution is

$$a_n = A(3^n) + B(-1)^n + \frac{3}{2}n(3^n)$$

Now we determine A, B from the initial data;

$$\begin{aligned} a_0 &= 2 = A + B \\ a_1 &= 3 = 3A - B + \frac{9}{2} \\ &\rightarrow A = \frac{1}{8}, \quad B = \frac{15}{8} \end{aligned}$$

Thus,

$$a_n = \frac{1}{8}(3^n) + \frac{15}{8}(-1)^n + \frac{3}{2}n(3^n)$$

(c) Calculate a_2 and a_3 using your answer from (b).

Solution:

$$\begin{aligned} a_2 &= \frac{1}{8}(9) + \frac{15}{8} + \frac{3}{2}(18) = 30 \\ a_3 &= \frac{1}{8}(27) - \frac{15}{8} + \frac{3}{2}(81) = 123 \end{aligned}$$

(3) (10) For $n \geq 1$ let a_n count the number of binary strings of length n, where there is no run of $1'^s$ of odd length. For example, when $n = 6$ we want to include the strings 110000 (which has a run of two $1'^s$ and a run of four $0'^s$) and 011110 (which has two runs of one 0 and one run of four $1'^s$), but we no not include either 100011 (which starts with a run of one 1) or 110111 (which ends with a run of three $1'^s$).

(a) Determine a_1, a_2, a_3, a_4.

Solution:

$$\begin{aligned} a_1 &= 1 \ \{0\} \\ a_2 &= 2 \ \{00, \ 11\} \\ a_3 &= 3 \ \{000, \ 011, \ 110\} \\ a_4 &= 5 \ \{0000, \ 0011, \ 0110, \ 1100, \ 1111\} \end{aligned}$$

(b) Find a recurrence relation for a_n (do not solve it).

Solution: Consider the n^{th} bit of a string of length n.

(i) if it is 0, then the preceding $n-1$ bits can occur in a_{n-1} ways

(ii) if it is 1, then the $(n-1)^{st}$ bit must also be a 1, and so the preceding $(n-2)$ bits can occur in a_{n-2} ways.

These cases cover all possibilities and do not overlap, and so

$$a_n = a_{n-1} + a_{n-2}$$

(4) (10) (a) Give an example of one loop-free graph with 5 vertices that has all of the following properties (indicate the features on your graph that satisfy these properties);

(i) There are no isolated vertices.

(ii) There is exactly one (distinct) cycle of length 4.

(iii) There is an edge such that if it is removed then the resulting graph has exactly two components.

(b) Give an example of a graph such that there is a trail between two distinct vertices that is not a path. (Recall that a trail is a walk where no edge is repeated, and a path is a walk where no vertex is repeated). Is it possible to have a path that is not a trail? Why or why not?

It is not possible to have a path that is not a trail, because if it is not a trail then one edge must be used more than once, and so the incident vertices to this edge would be used more than once.