$$
\begin{aligned}
(1+x)^n &= \left(\begin{array}{c} n \\ 0 \end{array}\right) + \left(\begin{array}{c} n \\ 1 \end{array}\right) x + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^2 + \cdots + \left(\begin{array}{c} n \\ n \end{array}\right) x^n \\
\frac{(1-x^{n+1})}{(1-x)} &= 1 + x + x^2 + x^3 + \cdots + x^n \\
\frac{1}{(1-x)} &= 1 + x + x^2 + x^3 \cdots \; = \; \sum_{i=0}^{\infty} x^i \\
\frac{1}{(1+x)^n} &= \left(\begin{array}{c} -n \\ 0 \end{array}\right) + \left(\begin{array}{c} -n \\ 1 \end{array}\right) x + \left(\begin{array}{c} -n \\ 2 \end{array}\right) x^2 + \cdots \\
&= \sum_{i=0}^{\infty} \left(\begin{array}{c} -n \\ i \end{array}\right) x^i \\
&= 1 + (-1) \left(\begin{array}{c} n+1-1 \\ i \end{array}\right) x + (-1)^2 \left(\begin{array}{c} n+2-1 \\ 2 \end{array}\right) x^2 + \cdots \\
&= \sum_{i=0}^{\infty} (-1)^i \left(\begin{array}{c} n+i-1 \\ i \end{array}\right) x^i \\
\frac{1}{(1-x)^n} &= \left(\begin{array}{c} -n \\ 0 \end{array}\right) + \left(\begin{array}{c} -n \\ 1 \end{array}\right) (-x) + \left(\begin{array}{c} -n \\ 2 \end{array}\right) (-x)^2 + \cdots \\
&= \sum_{i=0}^{\infty} \left(\begin{array}{c} -n \\ i \end{array}\right) (-x)^i \\
&= 1 + (-1) \left(\begin{array}{c} n+1-1 \\ i \end{array}\right) (-x) + (-1)^2 \left(\begin{array}{c} n+2-1 \\ 2 \end{array}\right) (-x)^2 + \cdots \\
&= \sum_{i=0}^{\infty} \left(\begin{array}{c} n+i-1 \\ i \end{array}\right) x^i
\end{aligned}
$$

MACM 201 Test 2

March 8, 2006. 50 minutes

Total marks: 55. Marks are indicated by ( ).

(1) (15) Consider the following recurrence relation;

$$a_n + 6a_{n-1} + 14a_{n-2} + 16a_{n-3} + 8a_{n-4} = 0, \quad a_0 = -1, \;\; a_1 = 1, \;\; a_2 = 2, \;\; a_3 = 3$$

Write down the general solution and the equations that will determine the unknown coefficients, but do not solve for the coefficients.

(2) (10) Consider the following recurrence relation;

$$a_n - 2a_{n-1} - 3a_{n-2} = 2(3^n), \quad a_0 = 2, \quad a_1 = 3$$

(a) Use the recurrence relation to determine a_2 and a_3.

(b) Solve the recurrence relation.

(c) Calculate a_2 and a_3 using your answer from (b).

(3) (10) Solve the following recurrence relation using generating functions;

$$a_{n+2} + 3a_n + 1 \; = \; n^3 + 2^n, \quad a_0 = 1, \; a_2 = 2$$

(4) (10) For $n \geq 1$ let a_n count the number of binary strings of length n, where there is no run of $1'^s$ of odd length. For example, when $n = 6$ we want to include the strings 110000 (which has a run of two $1'^s$ and a run of four $0'^s$) and 011110 (which has two runs of one 0 and one run of four $1'^s$), but we no not include either 100011 (which starts with a run of one 1) or 110111 (which ends with a run of three $1'^s$).

(a) Determine a_1, a_2, a_3, a_4.

(b) Find a recurrence relation for a_n (do not solve it).

(5) (10) (a) Give an example of <u>one</u> loop-free graph with 5 vertices that has all of the following properties (indicate the features on your graph that satisfy these properties);

(i) There are no isolated vertices.
(ii) There is exactly one (distinct) cycle of length 4.
(iii) There is an edge such that if it is removed then the resulting graph has exactly two components.

(b) Give an example of a graph such that there is a trail between two distinct vertices that is not a path. (Recall that a trail is a walk where no edge is repeated, and a path is a walk where no vertex is repeated). Is it possible to have a path that is not a trail? Why or why not?