MACM 201 Test 1 - Solutions

(1) (15 marks) How many integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 = 120$$

$$0 \le x_1 \le 65, \quad -5 \le x_2 \le 10, \quad 2 \le x_3 \le 20, \quad 0 \le x_4$$

Solution:

The number of solutions is the same as the number of solutions to

$$x_1 + x_2 + x_3 + x_4 = 123$$

$$0 \le x_1 \le 65, \quad 0 \le x_2 \le 15, \quad 0 \le x_3 \le 18, \quad 0 \le x_4$$

Let c_1 be the condition $x_1 \ge 66$, c_2 be $x_2 \ge 16$, and c_3 be $x_3 \ge 19$.

$N(c_1)$: $x_1 + x_2 + x_3 + x_4 = 57$, $0 \le x_1, x_2, x_3, x_4$. So $N(c_1) = \binom{4+57-1}{57}$

$N(c_2)$: $x_1 + x_2 + x_3 + x_4 = 107$, $0 \le x_1, x_2, x_3, x_4$. So $N(c_2) = \binom{4+107-1}{107}$

$N(c_3)$: $x_1 + x_2 + x_3 + x_4 = 104$, $0 \le x_1, x_2, x_3, x_4$. So $N(c_3) = \binom{4+104-1}{104}$

$N(c_1c_2)$: $x_1 + x_2 + x_3 + x_4 = 41$, $0 \le x_1, x_2, x_3, x_4$. So $N(c_1c_2) = \binom{4+41-1}{41}$

$N(c_1c_3)$: $x_1 + x_2 + x_3 + x_4 = 38$, $0 \le x_1, x_2, x_3, x_4$. So $N(c_1c_3) = \binom{4+38-1}{38}$

$N(c_2c_3)$: $x_1 + x_2 + x_3 + x_4 = 88$, $0 \le x_1, x_2, x_3, x_4$. So $N(c_2c_3) = \binom{4+88-1}{88}$

$N(c_1, c_2, c_3)$: $x_1 + x_2 + x_3 + x_4 = 22$, $0 \le x_1, x_2, x_3, x_4$. So $N(c_1, c_2, c_3) = \binom{4+22-1}{22}$

N : $x_1 + x_2 + x_3 + x_4 = 123$, $0 \le x_1, x_2, x_3, x_4$. So $N = \binom{4+123-1}{123}$

Our answer is

$$N(\bar{c}_1, \bar{c}_2, \bar{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

(2) (10 marks) Determine the coefficient of x^{13} in the function $f(x)$ where

$$f(x) \;=\; x^2\left(\frac{2}{x} - 4x^2\right)^{-5} \;-\; \frac{(3+5x^8+x^{10})}{1+2x}$$

Solution:

$$\begin{aligned}\frac{x^2}{\left(\frac{2}{x}-4x^2\right)^5} &= \frac{x^2}{\left[\frac{2}{x}(1-2x^3)\right]^5}\\ &= \frac{x^7}{2^5(1-2x^3)^5}\\ &= \frac{x^7}{2^5}\sum_{i=0}^{\infty}\left(\begin{array}{c}5+i-1\\ i\end{array}\right)2^i x^{3i}\end{aligned}$$

We need the coefficient of x^6, i.e., $i = 2$.

$$\frac{(3+5x^8+x^{10})}{1+2x} = (3+5x^8+x^{10})(1-2x+4x^2-8x^3+\cdots+(-1)^n2^nx^n+\cdots)$$

The coefficient of x^{13} here is

$$3(-1)2^{13} + 5(-1)2^5 - 2^3$$

So the answer is;

$$2^{-3}\left(\begin{array}{c}6\\ 2\end{array}\right) + 3\cdot 2^{13} + 5\cdot 2^5 + 2^3$$

(3) (10 marks) An unlimited supply of red, blue, yellow, and green marbles is available.

(a) (5 marks) Write down the generating function that will answer this problem: How many ways can n marbles be selected so that there are an even number of red marbles, an odd number of blue marbles, at least 7 but no more than 15 yellow marbles, and the same number of green marbles as the sum of the red, blue, and yellow ones? What coefficient are we looking for?

Solution:

First of all, if n is odd then the answer is 0 because the number of balls chosen this way has to be even (it's twice the number of red, blue, and yellow balls chosen). Once the red, blue and yellow balls have been chosen, there is only one way to choose the green balls. So assume n is even; $n = 2k$. Then the generating function is

$$\overset{red}{(1+x^2+x^4+\cdots)}\overset{blue}{(x^3+x^5+\cdots)}\overset{yellow}{(x^7+x^8+\cdots+x^{15})} = \frac{1}{(1-x^2)}\,\frac{x^3}{(1-x^2)}\,\frac{x^7(1-x^9)}{(1-x)}$$

We want the coefficient of x^k.

(b) (3 marks) Determine the number of ways to select the marbles if $n = 22$.

Solution:

We want the coefficient of x^{11}. From the formula above for the generating function, there are two ways to produce a term with x^{11} : x^1, x^3, x^7, and x^0, x^3, x^8, corresponding to the choice of 1 red, 3 blue, 7 yellow, and 11 green balls, and 0 red, 3 blue, 8 yellow, and 11 green balls.

(c) (2 marks) How many ways are there to select the marbles if $n = 41$?

Solution: There are no ways to choose 41 balls this way since 41 is not even.

(4) (10 marks) Consider the following recurrence relation;

$$6a_{n+2} + a_{n+1} - a_n = 0, \quad n \geq 0, \quad a_0 = 2, a_1 = -3$$

(a) (2 marks) Write out the terms $a_1,\ a_1,\ a_2,\ a_3$.

Solution:

$a_{n+2} = (a_n - a_{n+1})/6$. So we have $a_0 = 2, a_1 = -3, a_2 = (a_0 - a_1)/6 = (2+3)/6 = 5/6, a_3 = (a_1 - a_2)/6 = (-3 - (5/6))/6 = -23/36$.

(b) (8 marks) Solve the recurrence recurrence relation and check your answer with the values you found in part (a). What is a_{100}?

Solution: The characteristic equation is $6r^2+r-1 = (2r+1)(3r-1)$. The roots are $r = -\frac{1}{2}, \frac{1}{3}$. The general solution is $a_n = k_1(-\frac{1}{2})^n + k_2(\frac{1}{3})^n$. We determine k_1, k_2 from the initial conditions;

$$\begin{aligned} a_0 = 2 : &\qquad k_1 + k_2 = 2 \\ a_1 = -3 : &\qquad k_1(-\frac{1}{2}) + k_2(\frac{1}{3}) = -3 \end{aligned}$$

We find that $k_1 = \frac{22}{5}, k_2 = -\frac{12}{5}$, and so the solution is $a_n = \frac{22}{5}(-\frac{1}{2})^n - \frac{12}{5}(\frac{1}{3})^n$. For $n = 2$ this gives $a_3 = \frac{22}{5}(\frac{1}{4}) - \frac{12}{5}(\frac{1}{9}) = 5/6$. For $n = 100$ we have $a_{100} = \frac{22}{5}(\frac{1}{2^{100}}) - \frac{12}{5}(\frac{1}{3^{100}})$.