**MACM 201 Test 1**

February 6, 2006. 50 minutes

Total marks: 45. Marks are indicated by ( ).

$$
\begin{aligned}
N(\bar{c}_1, \bar{c}_2\bar{c}_3\cdots\bar{c}_t) &= N - [N(c_1) + N(c_2) + \cdots + N(c_t)] \\
&\quad + [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + \cdots + N(c_2c_3) + \cdots + N(c_{t-1}c_t)] \\
&\quad - [N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + N(c_1c_3c_4) + \cdots \\
&\qquad + N(c_1c_3c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)] + \cdots + (-1)^t N(c_1c_2c_3\cdots c_t) \\
&= S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t
\end{aligned}
$$

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t$$

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t$$

If $n \in \mathbf{Z}^+$,

$$\binom{-n}{r} = \binom{n+r-1}{r}$$

For all $m, n \in \mathbf{Z}^+,\ a \in \mathbf{R}$,

$$
\begin{aligned}
(1+x)^n &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n \\
\frac{(1-x^{n+1})}{(1-x)} &= 1 + x + x^2 + x^3 + \cdots + x^n \\
\frac{1}{(1-x)} &= 1 + x + x^2 + x^3 \cdots = \sum_{i=0}^{\infty} x^i \\
\frac{1}{(1+x)^n} &= \binom{-n}{0} + \binom{-n}{1} x + \binom{-n}{2} x^2 + \cdots \\
&= \sum_{i=0}^{\infty} \binom{-n}{i} x^i \\
&= 1 + (-1)\binom{n+1-1}{i} x + (-1)^2 \binom{n+2-1}{2} x^2 + \cdots \\
&= \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i \\
\frac{1}{(1-x)^n} &= \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots \\
&= \sum_{i=0}^{\infty} \binom{-n}{i} (-x)^i \\
&= 1 + (-1)\binom{n+1-1}{i}(-x) + (-1)^2 \binom{n+2-1}{2}(-x)^2 + \cdots \\
&= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i
\end{aligned}
$$

**(1)** (15) **Do not use generating functions to solve this problem.** *Use the Principle of Inclusion and Exclusion (Chapter 8).*

How many integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 = 120$$

$$0 \le x_1 \le 65, \quad -5 \le x_2 \le 10, \quad 2 \le x_3 \le 20, \quad 0 \le x_4$$

**(2)** (10) Determine the coefficient of $x^{13}$ in the function $f(x)$ where

$$f(x) \;=\; x^2\left(\frac{2}{x} - 4x^2\right)^{-5} \;-\; \frac{(3+5x^8+x^{10})}{1+2x}$$

**(3)** (10) An unlimited supply of red, blue, yellow, and green marbles is available.

(a) Write down the generating function that will answer this problem: How many ways can $n$ marbles be selected so that there are an even number of red marbles, an odd number of blue marbles, at least 7 but no more than 15 yellow marbles, and the same number of green marbles as the sum of the red, blue, and yellow ones? (Remember, 0 is an even number.) What coefficient are we looking for?
<u>Simplify your answer using the formula on the front page.</u>

(b) Determine the number of ways to select the marbles if $n = 22$. Describe how many of each color are selected for these choices.

(c) How many ways are there to select the marbles if $n = 21$?

**(4)** (10) Consider the following recurrence relation;

$$6a_{n+2} + a_{n+1} - a_n = 0, \quad n \geq 0, \quad a_0 = 2, a_1 = -3$$

(a) Write out the terms $a_1,\ a_1,\ a_2,\ a_3$.

(b) Solve the recurrence recurrence relation and check your answer with the value of $a_3$ you found in part (a). What is $a_{100}$?