MACM 201 Final Examination, April 2006

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Aids allowed: nonprogrammable calculator and formula sheet (provided).

Marks for each question are indicated by [ ]. Total marks: 114.

Time allowed: 180 minutes. There are **12** questions.

NAME:

STUDENT NUMBER:

(1) [12] Use the Principle of Inclusion and Exclusion to answer this question.

A standard deck of 52 cards contains cards labeled A (Ace), 2, 3, …, 10, J (Jack), Q (Queen), and K (King). The <u>numbered cards</u> are those labeled 2,3,4,…,10. There are 4 suits; clubs, spades, hearts, and diamonds. Each labeled card appears once in each suite (so for example, there are 4 Aces; one each in clubs, spades, hearts, and diamonds). Thus, $52 = 13 \times 4$.

(a) Suppose a hand of 10 cards are dealt out (randomly) from the deck. What is the probability that this hand will contain at least 1 card from each of the numbered cards 2, 3, …, 10? (they can be from any suit)

C_i ; No cards numbered i, $2 \le i \le 10$

$N(c_i) = \binom{48}{10}$; $N(c_i c_j) = \binom{44}{10}$, … , $N(c_2, .., c_{10}) = \binom{16}{10}$

~~We want~~ $N(\bar{c}_2 \bar{c}_3 \cdots \bar{c}_{10}) = S_0 - S_1 + \cdots - S_9$

$$= \binom{52}{10} - \binom{9}{1}\binom{48}{10} + \binom{9}{2}\binom{44}{10} - \cdots + \binom{9}{8}\binom{20}{10} - \binom{9}{9}\binom{16}{10}$$

Prob is $N(\bar{c}_2 \cdots \bar{c}_{10}) \Big/ \binom{52}{10}$

continued →

(b) What is the probability that this hand will contain at exactly one missing numbered card (i.e., for example, no card numbered 3)?

$$E_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 - \cdots - \binom{9}{8} S_9 \Big/ \binom{52}{10}$$

3

(c) What is the probability that this hand will contain at least 4 missing numbered cards?

$$L_4 = S_4 - \binom{4}{3} S_5 + \binom{5}{3} S_6 - \binom{6}{3} S_7 + \binom{7}{3} S_8 - \binom{8}{3} S_9 \Big/ \binom{52}{10}$$

3

(2) [12] Solve the following recurrence relation using generating functions.

$$a_{n+1} + 3a_n + 1 = n^3 + 2^n, \quad a_0 = 1$$

Check your answer using a_1.

$$\sum_0^\infty a_{n+1} x^{n+1} + 3\sum_0^\infty a_n x^{n+1} + \sum_0^\infty x^{n+1} = \sum_0^\infty (n^3+2^n) x^{n+1}$$

$$\left(f(x) - a_0\right) + 3x\, f(x) = x\sum_0^\infty (n^3+2^n-1) x^n$$

$$f(x) = \underbrace{\frac{x}{1+3x} \sum (n^3+2^n-1)x^n}_{\mathrm{I}} + \underbrace{\frac{1}{1+3x}}_{\mathrm{II}}$$

$$\frac{1}{1+3x} = \left(1 - 3x + 3^2x^2 - 3^3x^3 + 3^4x^4 - \dots + (-1)^n 3^n x^n + \cdots\right)$$

$$\frac{x}{1+3x} = \left(x - 3x^2 + 3^2x^3 - 3^3x^4 + 3^4x^5 - \dots + (-1)^n 3^n x^{n+1} + \cdots\right)$$

or partial fractions

Need coeff of x^n ;

I ; $\left((n-1)^3 + 2^{(n-1)} - 1\right) - 3\left((n-2)^3 + 2^{(n-2)} - 1\right) + 3^2\left((n-3)^3 + 2^{n-3} - 1\right) - \cdots$

II ; $(-1)^n 3^n$

(3) [12] Solve the following recurrence relation <u>without</u> using generating functions.

$$a_{n+2} - 6a_{n+1} + 9a_n = 2 + 2(3^n), \quad a_0 = -2,\ a_1 = 2$$

Check your answer using a_2.

$\therefore$ Char poly: $r^2 - 6r + 9 = (r-3)^2$; $r = 3$

$\therefore$ $a_n^h = A3^n + Bn3^n$

$\therefore$ $a_n^p = C + \underbrace{Dn3^n}_{\text{don't need}} + En^2 3^n$

sub into relation: . . . .

system: $C - 6C + 9C = 2$

$\vdots$

$9D + 36E - 18D - 36E + 9D = 2$

$\left.\right\}$ $C = \frac{1}{2}$, $E = \frac{1}{9}$

$\rightarrow a_n^p = \frac{1}{2} + \frac{1}{9}n^2 3^n$

Now determine A, B;

$n=0$; $a_0^p + a_0^h = \frac{1}{2} + A = -2 \rightarrow \boxed{A = -\frac{5}{2}}$

$n=1$; $a_1^p + a_1^h = \frac{1}{2} + \frac{1}{3} - \frac{15}{2} + 3B = 2 \rightarrow \boxed{B = {}^{26}/_{27}}$

Check; $a_2 = \overset{12}{6a_1} - \overset{+18}{9a_0} + 2 + 2 = 3\cancel{4}$

(4) [8] Recall that d_n is the number of derangements of $1, 2, \ldots, n$, i.e., how many ways the numbers $1, 2, \ldots, n$ can be arranged so that 1 is not in the first position, 2 is not in the second position, $\ldots$, n is not in the n^{th} position.

If $n > 2$, find a recurrence relation satisfied by d_n.

each der; '1' is placed in pos. i, $2 \le i \le n$

Then (A) 'i' is in pos 1; remaining $n-2$ integers deranged d_{n-2} ways
have $(n-1)$ choices for i $\rightsquigarrow$ $(n-1)d_{n-2}$.

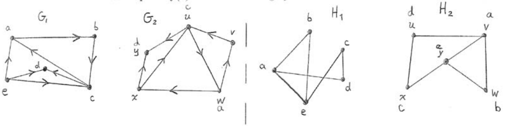
(B) 'i' not in pos 1 (or i):

$\vdots$

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \qquad d_2 = 1, \; d_3 = 2$$

67+
29

(5) [8] Consider the two pairs of graphs below; G_1, G_2 and H_1, H_2. Is G_1 and G_2 isomorphic? Is H_1 and H_2 isomorphic? If not, explain why not. If so, exhibit an isomorphism $f(v)$ between the two graphs.



No $\deg(c) = 4$; $\deg(u) = 4$

So must have $c \to u$

Also, must have $d \to y$

This forces $a \to w$,

but $\text{indeg}(a) = 2 \neq \text{indeg}(w)$

Yes

$a \to v$

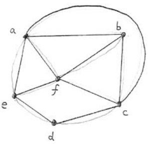
$e \to y$

$b \to w$

$c \to x$

$d \to u$

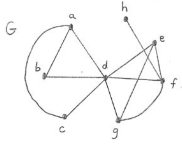
(6) [6] Find an Euler **trail** and a Hamiltonian **cycle** in the graph below.

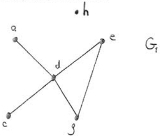


Euler: b - a - f - b - c - f - e - d - c - a - e

Ham: a - e - d - c - b - f - a

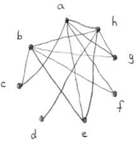
(7) [24] (a) Is the graph G_1 an induced subgraph of G? Explain your answer.





<u>No:</u> Vertex set of G_1 is $V_1 = \{a, c, d, e, g, h\}$
but missing edge $\{c, a\}$.

(b) Sketch the complement $\bar{G}$ of G. Begin by arranging the vertices of G along a circle, as given below.



continued →

(c) Is G bipartite? Explain. No;

$V_1 = \{a, h, e$

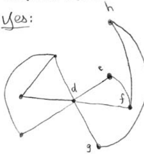
$V_2 = \{d, c, b, f, g$

but $\exists$ edge $\{d, b\}$!

impossible to partition

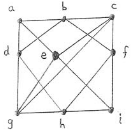
(d) Is G planar? Explain.

Yes:

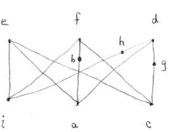


or: can argue that there is No homeomorphism to K_5 or $K_{3,3}$ because of degree of vertices.

(e) Is the graph below planar? Explain.



No:



(8) [4] Suppose T is a spanning tree of K_n, the complete graph on n vertices. Show that if an edge e from K_n, $e \notin T$, is added to T to produce the graph T', then T' contains a cycle.

Let $e = \{a, b\}$. The $\exists$ path in T; $a \to v_1 \to \cdots \to v_k \to b$

So now we have the cycle $a \to v_1 \to \cdots \to v_k \to b \to a$

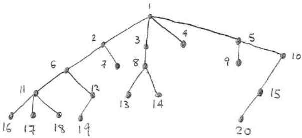
(9) [4] Prove or disprove: If G is a planar graph on $v \geq 4$ vertices, then $\bar{G}$ is not planar.

<u>False</u>:

G

$\bar{G}$

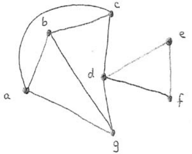
(10) [4] List the vertices in the tree below when they are visited in a preorder traversal and in a postorder traversal.

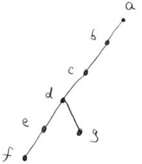


preorder 1, 2, 6, 11, 16, 17, 18, 12, 19, 7, 3, 8, 13, 14, 4, 5, 9, 10, 15, 20

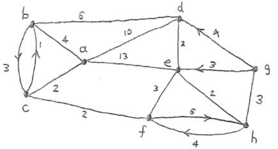
postorder 16, 17, 18, 11, 19, 12, 6, 7, 2, 13, 14, 8, 3, 4, 9, 20, 15, 10, 5, 1

(11) [4] Find the depth-first spanning tree for the graph below if the order of the vertices is alphabetical.

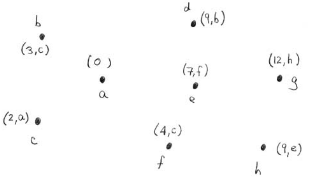




(12) [16] Consider the following graph G below;



(a) Use Dikjistra's Algorithm to find the lengths of the shortest paths from vertex a to all other vertices. What is the shortest path from a to d?



a-c-f-e-h

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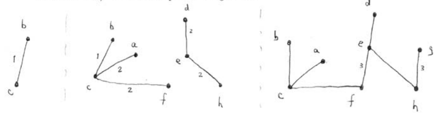
(b) Find a minimal spanning tree for G using Kruskal's Algorithm (assume all edges are undirected). Show the steps of the algorithm.

Kruskal:



$W = 15$

either way

Prim: $P = \{a, c, b, f, e, h, d, g\}$, $\{c, b, a, f, e, d, h, g\}$

$\{b, c, a, f, e, d, h, g\}$

Can start Prim's at any vertex...

(c) Find a minimal spanning tree for G using Prim's Algorithm (assume all edges are undirected). Show the steps of the Algorithm.